CHAPTER 1

Relations and Functions

Chapter Analysis of Previous Three Years' Board Exams

Number of Questions Asked in Exams

Topics	2018		2017		2016	
	Delhi	**All India**	**Delhi**	**All India**	**Delhi**	**All India**
Equivalence Relation	6 marks	6 marks				
Binary Operation	1 marks	1 marks	6 marks			
Invertible Function				6 marks	6 marks	6 marks

[TOPIC 1] Concept of Relations and Functions

Summary

Relation

Definition: If $(a, b) \in R$, we say that a is related to b under the relation R and we write it as aRb.

Domain of a relation: The set of first components of all the ordered pairs which belong to R is the domain of R.

Domain $(R) = \{a \in A : (a, b) \in R \forall\, b \in B\}$

Range of a relation: The set of second components of all the ordered pairs which belong to R is the domain of R.

Range of $R = \{b \in B : (a, b) \in R \forall\, a \in A\}$

Types of relations:

- **Empty relation:** Empty relation is the relation R from X to Y if no element of X is related to any element of Y, it is given by $R = \varphi \subset X \times Y$.

  For example, let $X = \{2, 4, 6\}$, $Y = \{8, 10, 12\}$

  $R = \{(a, b) : a \in X, b \in Y \text{ and } a + b \text{ is odd}\}$

  R is an empty relation.
- **Universal relation:** Universal relation is a relation R from X to Y if each element of X is related to every element of Y it is given by $R = X \times Y$.

  For example, let $X = \{x, y\}$, $Y = \{x, z\}$

  $R = \{(x, x), (y, z), (y, x), (y, z)\}$

  $R = X \times Y$, so relation R is a universal relation.
- **Reflexive relation:** Reflexive relation R in X is a relation with $(a, a) \in R \forall a \in X$.

  For example, let $X = \{x, y, z\}$ and relation R is given as

  $R = \{(x, x), (y, y), (z, z)\}$

  Here, R is a reflexive relation on X.
- **Symmetric relation:** Symmetric relation R in X is a relation satisfying $(a, b) \in R$ implies $(b, a) \in R$.

  For example, let $X = \{x, y z\}$ and relation R is given as

  $$R = \{(x, y), (y, x)\}$$

  Here, R is a symmetric relation on X.
- **Transitive relation:** Transitive relation R in X is a relation satisfying $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, b) \in R$.

  For example, let $X = \{x, y, z\}$ and relation R is given as

  $$R = \{(x, z), (z, y), (x, y)\}$$

  Here, R is a transitive relation on X.
- **Equivalence relation:** It is a relation R in X which is reflexive, symmetric and transitive.

  For example, let $X = \{x, y, z\}$ and relation R is given as

  $R = \{(x, y), (x, x), (y, x), (y, y), (z, z), (x, z), (z, x), (y, z)\}$

  Here, R is reflexive, symmetric and transitive. So R is an equivalence relation on X.
- Equivalence class $[a]$ containing $a \in X$ for an equivalence relation R in X is the subset of X containing all elements b related to a.

Function

Definition: A rule f which associates each element of a non-empty set A with a unique element of another non-empty set B is called a function.

Types of functions:

- **Injective function:** A function $f : X \to Y$ is one-one (or injective) if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \forall x_1, x_2 \in X$.
- **Surjective function:** A function $f : X \to Y$ is onto (or surjective) if given any $y \in Y$, $\exists x \in X$ such that $f(x) = y$.
- **Bijective function:** A function $f : X \to Y$ is one-one and onto (or bijective), if f is both one-one and onto.
- **Composite function:** The composition of functions $f : A \to B$ and $g : B \to C$ is the function $gof : A \to C$ given by $gof(x) = g(f(x)) \forall\, x \in A$
- **Invertible function:** A function $f : X \to Y$ is invertible if $\exists g : Y \to X$ such that $gof = I_X$ and $fog = I_Y$.

  A function $f : X \to Y$ is invertible if and only if f is one-one and onto.
- **Steps to find inverse of a function**

  Let $f(x) = y$ where $x \in X$ and $y \in Y$

  Solve $f(x) = y$ for x in terms of y.

  Now replace x with $f^{-1}(y)$ in the expression obtain from the above step.

  Finally to find the inverse function of $ff^{-1}(x)$ replace y with x in the expression obtained from the above step.

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

1 Mark Questions

1. State the reason for the relation R in the set {1, 2, 3} given by $R = \{(1, 2), (2, 1)\}$ not to be transitive.

[DELHI 2011]

2. If the function $f : R \to R$ be defined by $f(x) = 2x - 3$ and $g : R \to R$ by $g(x) = x^3 + 5$, then find the value of $(fog)^{-1}(x)$.

[ALL INDIA 2015]

3. If $R = \{(x, y) : x + 2y = 8\}$ is a relation on N, Write the range of R.

[ALL INDIA 2014]

4. $R \to R$ is defined by $f(x) = 3x + 2$, define f[f(x)].

[ALL INDIA 2011]

4 Marks Questions

5. Show that the function f in

$A = R - \left\{\frac{2}{3}\right\}$ defined as $f(x) = \frac{4x+3}{6x-4}$ is one-one and onto. Hence find f^{-1}.

[DELHI 2013]

6. Let $A = \{1, 2, 3, ...9\}$ and R be the relation in $A \times A$ defined by $(a, b) R (c, d)$.

If a + d = b + c for (a, b) (c, d) in $A \times A$, prove that R is an equivalence relation. Also obtain the equivalence class $[(2, 5)]$.

[DELHI 2014]

7. Let A = R – {3} and B = R – {1}. Consider the function f: A → B Defined by $f(x) = \left(\frac{x-2}{x-3}\right)$ show that f is one-one and also hence find f^{-1}.

[ALL INDIA 2012]

8. Consider $f : R^+ \to [4, \infty]$ given by $f(x) = x^2 + 4$.. Show that f is invertible with the inverse (f^{-1}) of f given by $f^{-1}(y) = \sqrt{y-4}$, where R^+ is the set of all non-negative real numbers.

[ALL INDIA 2011, 2013]

9. Let A = R – {3} and B = R – {1} consider the function $f : A \to B$ defined by $f(x) = \left(\frac{x-2}{x-3}\right)$. Show that f is one-one and onto and hence find $f^{-1}(x)$.

[DELHI 2012]

10. If the function $f : R \to R$ be given by $f(x)\ x^2 + 2$ and $g : R \to R$ be given by $g(x) = \frac{x}{x-1}$, $x \neq 1$, find fog and gof and hence find fog (2) and gof (–3).

[ALL INDIA 2014]

6 Marks Questions

11. Consider $f: R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with

$$f^{-1}(y) = \left(\frac{\sqrt{y+6}-1}{3}\right).$$

Hence find

(i) $f^{-1}(10)$

(ii) y if $f^{-1}(y) = \frac{4}{3}$, Where R_+ is the set of all non-negative real numbers.

[DELHI 2017]

12. Let N denote the set of all natural numbers and R be the relation on N × N defined by $(a, b) R (c, d)$ if $ad(b + c) = bc(a + d)$. Show that R is an equivalence relation.

[DELHI 2015]

13. Let $f : N \to N$ be a function defined as $f(x) = 9x^2 + 6x - 5$. Show that $f : N \to N$, where S is the range of, is invertible. Find the inverse of f and hence find $f^{-1}(43)$ and $f^{-1}(163)$

[DELHI 2016]

14. Consider $f : R - \left\{\frac{4}{3}\right\} \to R - \left\{\frac{4}{3}\right\}$ given by

$f(x) = \frac{4x+3}{3x+4}$. Show that f is objective. Find the inverse and hence find $f^{-1}(0)$ and x such that $f^{-1}(x) = 2$

[ALL INDIA 2017]

15. Let and $A = R \times R$ and * be a binary operation on A defined by $(a,b)*(c,d) = (a+c, b+d)$. Show that * Find the identity element for * on A. Also find the inverse of every element $(a, b) \in A$.

[ALL INDIA 2016]

16. Show that the function $f : \text{R} \to \text{R}$ defined by

$f(x) = \dfrac{x}{x^2+1}, \forall \text{x} \in \mathbb{R}$ is neither one-one nor onto. Also $\text{g} : \text{R} \to \text{R}$ if is defined as $g(x) = 2x - 1$, find fog (x).

[DELHI 2018]

17. Let $A = \{x \in \mathbf{Z} : 0 \le x \le 12\}$, Show $\{(a, b) : |a - b|$ is divisible by 4$\}$ that is an equivalence relation. Find the set of all elements related to 1. Also write the equivalence class [2].

[DELHI 2018]

Solutions

1. Let R = {1, 2, 3}

As $(1,2) \in R$ and $(2,1) \in R$

But $(1,1) \notin R$ [1]

So, R is not transitive.

2. Let $y = (\text{fog})(x)$

Let $y = \text{h}(\text{x})$

$= \text{f}[\text{g}(\text{x})] = \text{f}(\text{x}^3 + 5)$ [½]

$= 2\text{x}^3 + 7$

Thus, $x = \sqrt[3]{\dfrac{y-7}{2}} = h^{-1}(y)$

$\Rightarrow (\text{fog})^{-1}_{(x)} = \sqrt[3]{\dfrac{y-7}{2}}$ [½]

3. R = $\{(\text{x, y}) : \text{x} + 2\text{y} = 8\}$ is a relation on N

Then we can say 2y = 8 – x

$\text{y} = 4 - \dfrac{\text{x}}{2}$

so we can put the value of x, x = 2, 4, 6 only

we get y = 3 at x = 2

we get y = 2 at x = 4

we get y = 1 at x = 6 [1]

so range = {1, 2, 3}

4. We have

f(x) = 3x + 2

Now, $\text{f}[\text{f}(\text{x})] = 3(3\text{x}+2)+2$

= 9x +6 + 2

= 9x + 8 [1]

5. Given $f(x) = \dfrac{4x+3}{6x-4}$

For f is one-one

$Let\, x_1, x_2 \in R$

$f(x_1) = f(x_2)$

$\dfrac{4x_1+3}{6x_1-4} = \dfrac{4x_2+3}{6x_2-4}$ [1]

$\therefore (4x_1+3)(6x_2-4) = (4x_2+3)(6x_1-4)$

$\therefore 24x_1x_2 - 16x_1 + 18x_2 - 12$

$\therefore -16x_1 - 18x_1 = -16x_2 - 18x_2$

$\therefore -34x_1 = -34x_2 \quad \Rightarrow x_1 = x_2$

f is one-one

Let y be any element of R.

$y = f(x) \quad \Rightarrow y = \dfrac{4x+3}{6x-4}$ [1]

$\Rightarrow 6xy - 4y = 4x + 3$

$\Rightarrow -4y - 3 = 4x - 6xy$

$\Rightarrow -4y - 3 = (4-6y)x$

$\Rightarrow \dfrac{-(4y+3)}{-(6y-4)} = x$

$\Rightarrow x = \dfrac{4y+3}{6xy-4}$

$f(x) = \dfrac{4x+3}{6x-4}$

$f\left(\dfrac{4y+3}{6y-4}\right) = \dfrac{4\left(\dfrac{4y+3}{6y-4}\right)+3}{6\left(\dfrac{4y+3}{6y-4}\right)-4}$ [1]

$$= \frac{\frac{16y+12+18y-12}{6y-4}}{\frac{24y+18-24y+16}{6y-4}}$$

$$= \frac{34y}{34} = y$$

$$F^{-1}(y) = \frac{4y+3}{6y-4};\ F^{-1}(x) = \frac{4x+3}{6x-4}$$ [1]

6. **Reflexive:** Let $a, b \in N$

$\because a+b = b+a \Rightarrow (a,b)\,R(a,b)$ [1]

$\therefore$ R is reflexive

Symmetric: Let a, b, c, d $\in$ N

$\because (a,b)\,R(c,d) \Rightarrow a+d = b+c$

$\Rightarrow b+c = a+d \Rightarrow c+b = d+a$ [1]

$\Rightarrow (c,d)\,R(a,b)$

$\therefore$ R is symmetric

Transitive: $a, b, c, d, e, f \in N$

Let $(a,b)\,R(c,d)\, and (c,d)\,R(e,f)$

$\Rightarrow a+d = b+c \text{ and } c+f = d+e$

$\Rightarrow a+d+c+f = b+c+d+e$ [1]

$\Rightarrow a+f = b+e \Rightarrow (a,b)\,R(e,f)$

$\therefore$ R is transitive.

$\because$ R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

For equivalence class. [1]

7. Given: For f is one-one

Let $x_1, x_2 \in R$

$f(x_1) = f(x_2)$ [1]

$$\therefore \frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3}$$

$\therefore (x_1-2)(x_2-3) = (x_1-3)(x_2-2)$

$\therefore x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$

$\therefore -3x_1 + 2x_1 = -3x_2 + 2x_2$

$-x_1 = -x_2 \Rightarrow x_1 = x_2$

$\therefore f$ is one - one [1]

Now for f is onto

Let y be any element of R

$\therefore y = f(x)$

$$\therefore y = \frac{x-2}{x-3}$$

$$\therefore y = \frac{x-2-1+1}{x-3}$$

$$\therefore y = \frac{(x-3)}{x-3} + \frac{1}{x-3}$$

$$\therefore y-1 = \frac{1}{x-3}$$

$$\therefore x-3 = \frac{1}{y-1}$$

$$\therefore x = \frac{1}{y-1} + 3$$

$\therefore (y-1)(x-3) = 1$

$$\therefore x = \frac{1+3y-3}{y-1}$$

$$f(x) = \frac{x-2}{x-3}$$

$$f\left(\frac{3y-2}{y-1}\right) = \frac{\frac{3y-2}{y-1}-2}{\frac{3y-2}{y-1}-3}$$ [1]

$$= \frac{\frac{3y-2-2y+2}{y-1}}{\frac{3y-2-3y+3}{y-1}}$$

$$= \frac{y}{1} = y$$

Thus, f is onto

$$f^{-1}(y) = \frac{3y-2}{y-1};$$

$$f^{-1}(x) = \frac{3x-2}{x-1}$$ [1]

8. $f(x) = x^2 + 4$

Let $f(x) = f(y)$

$x^2 + 4 = y^2 + 4$

$x^2 = y^2$

$\therefore x = y$ [1]

As $x = y \in R^+$

Hence, f is a one-one function.

For $y \in [4, \infty)$,

Let $y = x^2 + 4$ [1]

$\therefore x^2 = y - 4 \geq 0$

$\therefore x = \sqrt{y-4} \geq 0$

Thus, for any $y \in R$.

There exist $x = \sqrt{y-4} \in R$ such that

$f(x) = f\left(\sqrt{y-4}\right)$

$= \left(\sqrt{y-4}\right)^2 + 4$

$= y - 4 + 4$

$= y$

Hence f is onto.

Therefore, f is one-one and onto. [1]

Now, calculate f^{-1}.

Let, $g(y) = \sqrt{y-4}$

(go f) (x)

$= g\left[f(x)\right]$

$= g(x^2 + 4)$

$= \sqrt{(x^2+4) - 4} = x$

(fog) (y)

$= f\left[g(y)\right]$

$= f\left(\sqrt{y-4}\right)$

$= \left(\sqrt{y-4}\right)^2 + 4$

$= y - 4 + 4 = y$

$\therefore gof = fog$ [1]

Hence, f is invertible

$f^{-1}(y) = g(y) = \sqrt{y-4}$

9. Now, $f(x_1) = f(x_2)$ [1]

$\Rightarrow \dfrac{x_1 - 2}{x_1 - 3} = \dfrac{x_2 - 2}{x_2 - 3}$

After solving

$-x_1 = -x_2$

$\Rightarrow x_1 = x_2$

Hence f is one-one function. [1]

For Onto

$x = \dfrac{3y-2}{y-1}$ [1]

Hence f is onto function.

Thus f is one-one onto function.

If f^{-1} is inverse function of f then

$f^{-1}(y) = \dfrac{3y-2}{y-1}$ [1]

10. $f = R \times R$

$f(x) = x^2 + 2$

$g : R \times R$

$g(x) = \dfrac{x}{x-1}, x \neq 1$

fog = $f(g)x))$

$= f\left(\dfrac{x}{x-1}\right) = \left(\dfrac{x}{x-1}\right)^2 + 2$

$= \dfrac{x^2}{(x-1)^2} + 2$

$= \dfrac{x^2 + 2(x-1)^2}{(x-1)^2}$

$= \dfrac{x^2 + 2x^2 - 4x + 2}{(x-1)^2}$

$= \dfrac{3x^2 - 4x + 2}{(x-1)^2}$ [1]

gof = g(f(x))

$= g(x^2 + 2)$

$= \dfrac{(x^2+2)}{(x^2+2) - 1}$

$= \dfrac{x^2+2}{x^2+1} = 1 + \dfrac{1}{x^2+1}$ [1]

$$\therefore \text{fog}(2)=\frac{3(2)^2-4(2)+2}{(2-1)^2}=6$$ [1]

$$\text{gof}(-3)=1+\frac{1}{(-3)^2+1}=\frac{11}{10}=1\frac{1}{10}$$ [1]

11. $f(x) = 9x^2 + 6x - 5$.

Let y be an arbitrary element of range f. [1]

Then $y = 9x^2 + 6x - 5$

$$\Rightarrow y=9\left(x^2+\frac{6}{9}x-\frac{5}{9}\right)$$

$$\Rightarrow y=9\left[x^2+\frac{2}{3}x+\left(\frac{1}{3}\right)^2-\frac{5}{9}-\left(\frac{1}{3}\right)^2\right]$$ [1]

$$\Rightarrow y=9\left[\left(x+\frac{1}{3}\right)^2-\frac{6}{9}\right]$$ ...(i)

$$\Rightarrow \frac{y}{9}=\left(x+\frac{1}{3}\right)^2-\frac{6}{9}$$

$$\Rightarrow \frac{y}{9}+\frac{6}{9}=\left(x+\frac{1}{3}\right)^2$$

Taking square root on both sides

$$\left(x+\frac{1}{3}\right)=\frac{\sqrt{y+6}}{3}$$ [1]

$$x=\frac{\sqrt{y+6}}{3}-\frac{1}{3}$$

$$=\frac{\sqrt{y+6}-1}{3}$$

Or $g(y)=\dfrac{\sqrt{y+6}-1}{3}$

$$\Rightarrow gof(x)=g(f(x))=g(9x^2+6x-5)$$ [1]

$$=\frac{\sqrt{9x^2+6x-5+6}-1}{3}$$

$$=\frac{\sqrt{9x^2+6x+1}-1}{3}$$

$$=\frac{\sqrt{(3x+1)^2}-1}{3}=\frac{3x+1-1}{3}$$

$$=\frac{3x}{3}=x$$ ...(ii)

And fog $(y) = f(g(y))$

$$=f\left(\frac{\sqrt{y+6}-1}{3}\right)$$

$$=9\left[\left(\frac{\sqrt{y+6}-1}{3}+\frac{1}{3}\right)^2-\frac{6}{9}\right]$$ .... from (i)

$$=9\left[\left(\frac{\sqrt{y+6}-1+1}{3}\right)^2-\frac{6}{9}\right]$$

$$=9\left[\left(\frac{\sqrt{y+6}}{3}\right)^2-\frac{6}{9}\right]$$

$$=9\left[\frac{y+6}{9}-\frac{6}{9}\right]$$

$$=\frac{9(y+6)}{9}-\frac{6\times 9}{9}$$

$= y + 6 - 6 = y$ ...(iii)

From (ii) and (iii), f is invertible and $f^{-1} = g$.

$$f^{-1}(y)=\frac{\sqrt{y+6}-1}{3}, y\in S$$

(i) $f^{-1}(10)=\dfrac{\sqrt{10+6}-1}{3}$

$$\frac{\sqrt{16}-1}{3}=\frac{4-1}{3}=\frac{3}{3}=1$$ [1]

(ii) $f^{-1}(y)=\dfrac{\sqrt{y+6}-1}{3}$

$$\therefore \frac{4}{3}=\frac{\sqrt{y+6}-1}{3}$$

$$\therefore 4+1=\sqrt{y+6}$$

$$\therefore 5=\sqrt{y+6}$$ [1]

Squaring on both sides we get

$25 = y + 6$

$\therefore\ y = 19$

12. (i) Reflexive: Clearly $(a, b)\ R\ (a, b)$s

Since $ab(b+a) = ba(a+b)\ \forall (a,b) \in N \times N$ [1]

$\therefore$ R is reflexive.

(ii) Symmetric: Let $(a,b)\ R(c,d)$

$\Rightarrow ad(b+c) = bc(d+a)$

$\Rightarrow cb(d+a) = da(c+b)$ [1]

$\Rightarrow (c,d)\ R(a,b)\ \forall (a,b),(c,d) \in N \times N$

$\therefore$ R is symmetric.

(iii) Let $(a,b)\ R(c,d)\ and (c,d)\ R(e,f)$

for $a,b,c,d,e,f \in N$ [1]

$\therefore ad(b+c) = bc(a+d)\ and\ cf(d+e) = de(c+f)$

$\frac{b+c}{bc} = \frac{a+d}{ad}$	$\frac{d+e}{de} = \frac{c+f}{cf}$
$\frac{b}{bc} + \frac{c}{bc} = \frac{a}{ad} + \frac{d}{ad}$	$\frac{d}{de} + \frac{e}{de} = \frac{c}{cf} + \frac{f}{cf}$
$\frac{1}{c} + \frac{1}{b} = \frac{1}{d} + \frac{1}{a}$ .....(*i*)	$\frac{1}{e} + \frac{1}{d} = \frac{1}{f} + \frac{1}{c}$ .....(*ii*)

[1]

Adding (i) and (ii), we have

$$\frac{1}{c} + \frac{1}{b} + \frac{1}{e} + \frac{1}{d} = \frac{1}{d} + \frac{1}{a} + \frac{1}{f} + \frac{1}{c}$$ [1]

$$\frac{e+b}{be} = \frac{f+a}{af}$$

$$af(b+e) = be(a+f)$$

Hence, $(a,d)\ R(e,f)$

$\therefore$ R is transitive

$\because$ R is reflexive, symmetric and transitive, [1]
Hence R is an equivalence relation.

13. $f(x) = 9x^2 + 6x - 5$

Let y be an arbitrary element of range f. [1]

Then $y = 9x^2 + 6x - 5$

$$\Rightarrow y = 9\left(x^2 + \frac{6}{9}x - \frac{5}{9}\right)$$

$$\Rightarrow y = 9\left[x^2 + \frac{2}{3}x + \left(\frac{1}{3}\right)^2 - \frac{5}{9} - \left(\frac{1}{3}\right)^2\right]$$

$$\Rightarrow y = 9\left[\left(x + \frac{1}{3}\right)^2 - \frac{6}{9}\right] \quad \text{...(i)}$$ [1]

$$\Rightarrow \frac{y}{9} = \left(x + \frac{1}{3}\right)^2 - \frac{6}{9}$$

$$\Rightarrow \frac{y}{9} + \frac{6}{9} = \left(x + \frac{1}{3}\right)^2$$

Taking square root on both sides

$$\left(x + \frac{1}{3}\right) = \frac{\sqrt{y+6}}{3}$$

$$x = \frac{\sqrt{y+6}}{3} - \frac{1}{3} = \frac{\sqrt{y+6} - 1}{3}$$

Or $g(y) = \dfrac{\sqrt{y+6} - 1}{3}$

$$\Rightarrow gof(x) = g(f(x)) = g(9x^2 + 6x - 5)$$ [1]

$$= \frac{\sqrt{9x^2 + 6x - 5 + 6} - 1}{3}$$

$$= \frac{\sqrt{9x^2 + 6x + 1} - 1}{3}$$

$$= \frac{\sqrt{(3x+1)^2} - 1}{3}$$

$$= \frac{3x + 1 - 1}{3}$$

$$= \frac{3x}{3} = x \quad \text{....(ii)}$$

And $fog(y) = f(g(y))$

$$= f\left(\frac{\sqrt{y+6} - 1}{3}\right)$$

$$= 9\left[\left(\frac{\sqrt{y+6} - 1}{3} + \frac{1}{3}\right)^2 - \frac{6}{9}\right] \quad \text{(from (i))}$$

$$= 9\left[\left(\frac{\sqrt{y+6} - 1 + 1}{3}\right)^2 - \frac{6}{9}\right]$$ [1]

$= 9\left[\left(\frac{\sqrt{y+6}}{3}\right)^2 - \frac{6}{9}\right]$

$= 9\left[\frac{y+6}{9} - \frac{6}{9}\right]$

$= \frac{9(y+6)}{9} - \frac{6\times 9}{9}$

$= y + 6 - 6 = y$ ....(iii)

From (ii) and (iii), invertible and $f^{-1} = g$.

$f^{-1}(y) = \frac{\sqrt{y+6-1}}{3}, y \in S$ [1]

or $f^{-1}(43) = \frac{\sqrt{43+6}-1}{3}$

$= \frac{7-1}{3} = \frac{6}{3} = 2$

or $f^{-1}(163) = \frac{\sqrt{163+6}-1}{3}$

$= \frac{13-1}{3} = \frac{12}{3} = 4$ [1]

14. $f(x) = \frac{4x+3}{3x+4}$ $x \in R - \left\{-\frac{4}{3}\right\}$

F is one-one

Let $x_1, x_2 \in R - \left\{-\frac{4}{3}\right\}$ and $f(x_1) = f(x_2)$ [1]

$\Rightarrow \frac{4x_1+3}{3x_1+4} = \frac{4x_2+3}{3x_2+4}$

$\Rightarrow 12x_1x_2 + 16x_1 + 9x_2 + 12$

$= 12x_1x_2 + 9x_1 + 16x_2 + 12$

$\Rightarrow 7x_1 = 7x_2 \Rightarrow x_1 = x_2$

Thus f is one-one

Now, f is onto

Let $k \in R - \left\{\frac{4}{3}\right\}$ be any number. [1]

$f(x) = k$

$\Rightarrow \frac{4x+3}{3x+4} = k$

$\Rightarrow 4x + 3 = 3kx + 4k$

$\Rightarrow x = \frac{4k-3}{4-3k}$

Also $\frac{4k-3}{4-3k} = -\frac{4}{3}$

Implies –9 = – 16 (Which is impossible)

$\therefore f\left(\frac{4k-3}{4-3k}\right) = k$, i.e. f is onto [1]

∴ The function f is invertible. i.e. f^{-1} exist inverse of *f*.

Let $f^{-1}(x) = k$

$f(k) = x$

$\Rightarrow \frac{4k+3}{3k+4} = x$ [1]

$\Rightarrow k = \frac{(4x-3)}{4-3x}$

$\therefore f^{-1}(x) = \frac{4x+3}{4-3x}$,

$x \in R - \left\{-\frac{4}{3}\right\}$

$f^{-1}(0) = -\frac{3}{4}$

When $f^{-1}(x) = 2$ [1]

$\Rightarrow \frac{(4x-3)}{4-3x} = 2$

$\Rightarrow 4x - 3 = 8 - 6x$

$\Rightarrow 10x = 11$

$\Rightarrow x = \frac{11}{10}$ [1]

15. $A = R \times R$

$(a, b) * (c, d) = (a + c, b + d)$

Commutataive:

Lete $(a, b), (c, d) \in A$

$= (c + a, d + b)$

$= (c, d) * (a, b)$ $\forall$ (a, b) (c, d) A [1½]

where * is commutative

Associative:

Let $(a,b),(c,d),(e,f) \in A$

$((a,b)*(c,d))*(e,f) = (a+c,b+d)*(e,f)$

$= (a+c+e, b+d+f)$

$= (a+(c+e), b+(d+f))$

$= (a,b)*(c+e, d+f)$

$= (a,b)*((c,d)(e,f))$

$\forall (a,b),(c,d),(e,f) \in A$ [1½]

where * is associative

Identity element:

Let $(e_1, e_2) \in A$

is identity element for * operation by definition

$\Rightarrow (a,b)*(e_1,e_2) = (a,b)$

$\Rightarrow (a+e_1, b+e_2) = (a,b)$

$a+e_1 = a, b+e_2 = b$

$\Rightarrow e_1 = 0, e_2 = 0$

$\Rightarrow (0,0) \in A$

$\Rightarrow$ (0, 0) is identity element for * [1½]

Inverse:

Let $(b_1, b_2) \in A$ is inverse of element $(a,b) \in A$ then by definition

$(a,b)*(b_1,b_2) = (0,0)$

$(a+b_1, b+b_2) = (0,0)$

$\Rightarrow a+b_1 = 0, b+b_2 = 0$

$\Rightarrow (-a,-b) \in A$ is inverse of every element $(a, b) \in A$. [1½]

16. Given defined by $f(x) = \dfrac{x}{x^2+1}, \forall x \in \mathbb{R}$

Checking for One - One (Injectivity):

Let $a, b \in R$ Then,

$f(a) = f(b)$ [1]

$$\Rightarrow \frac{a}{a^2+1} = \frac{b}{b^2+1}$$

$a(b^2+1) = b(a^2+1)$

$ab^2 + a = ba^2 + b$

$ab^2 + a - ba^2 - b = 0$

$ab(b-a) - 1(b-a=0)$

$b-a=0$ or $ab-1=0$

$b=a$ or $ab=1$

$\Rightarrow a = \dfrac{1}{b}$

Hence, $f(x)$ is not one-one or Injective. [1]

Onto (Surjectivity):

Let $y \in R$

Then $y = f(x)$

$\Rightarrow y = \dfrac{x}{x^2+1}$ [1]

$x^2 y + y = x$

$x^2 y + y - x = 0$

To solve for x we use the quadratic formula.

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ where

$a = y$, $b = -1$ and $c = y$

$x = \dfrac{1 \pm \sqrt{1-4y^2}}{2y}$

$1 - 4y^2 \geq 0$

$\Rightarrow 4y^2 \leq 1$

$\Rightarrow y^2 \leq \dfrac{1}{4}$

$y^2 \leq \left(\dfrac{1}{2}\right)^2$

$y \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$

$\left[x^2 \leq a^2 \text{ then } -a \leq x \leq a\right]$ [1]

$\Rightarrow$ Range is not equal to codomain.

Thus, $f(x)$ is not onto.

Given $f(x) = \frac{x}{x^2+1}$, $g(x) = 2x - 1$

$fog(x) = f(g(x))$

$fog(x) = f(2x-1)$

$fog(x) = \frac{2x-1}{(2x-1)^2+1}$

$fog(x) = \frac{2x-1}{4x^2+1-4x+1}$

$fog(x) = \frac{2x-1}{4x^2-4x+2}$

$fog(x) = \frac{2x-1}{2(2x^2-2x+1)}$ [2]

17. $A = \{x \in \mathbf{Z} : 0 \le x \le 12\}$,

$\{(a, b) : |a - b|$ is divisible by 4$\}$

Equivalence relation:

Let $x \in A$

$|x - x| = 0$. and 0 is divisible by 4. [1]

$\Rightarrow |x - x| \in \mathbb{R}$

$\therefore$ R is reflexive.

Let $(x, y) \in A, |x - y|$ is divisible by 4.

$\Rightarrow |x - y| = 4k, \quad k \in \mathbb{Z}$

$\Rightarrow |y - x| = 4k, \quad k \in \mathbb{Z}$

$\Rightarrow (x, y) \in R$

$\therefore$ R is symmetric. [1]

Transitive:

Let $x, y, z \in A, |x - y|$ is divisible by 4.

$\Rightarrow |x - y| = 4k, \quad k \in \mathbb{Z}$ [1]

$\Rightarrow x - y = 4k$ or $x - y = -4k$

$x - y = \pm 4k$ (1)

And assume $|y - z|$ is divisible 4.

$\Rightarrow |y - z| = 4n, \quad n \in \mathbb{Z}$

$\Rightarrow y - z = 4n$ or $y - z = -4n$

$y - z = \pm 4n$ (2)

Adding (1) and (2) we get.

$(x - y) + (y - z) = (\pm 4k) + (\pm 4n)$

$x - z = \pm 4(k + n)$

$\Rightarrow x - z$ is divisible by 4.

$/x - z/$ is divisible by 4.

$\Rightarrow (x, z) \in R$

$\therefore$ It is transitive. [1]

As all three properties are satisfied so given relation R is an equivalence relation.

Let a be an element of A such that $(a, 1) \in R$

$|a - 1|$ is divisible by 4.

$0 \le a \le 12$

$|a - 1| = 0, 4, 8, 12$

$a - 1 = 0, 4, 8, 12$

$a - 1 = 0$

$a = 1$

$a - 1 = 4$

$a = 5$

$a = -1 = 8$

$a = 9$

$a - 1 = 12$

$a = 13$ which is not possible [1]

{1, 5, 9} is the set of elements of A related to 1.

To find the equivalence class [2]

Let a be an element of A such that $(a, 2) \in R$

$|a - 2|$ is divisible by 4.

$|a - 2| = 4n \qquad n \in \mathbb{Z}$

$|a - 2| = 0, 4, 8, 12$

$a - 1 = 0, 4, 8, 12$

$a - 2 = 0$

$a = 2$

$a - 2 = 4$

$a = 6$

$a - 2 = 8$

$a = 10$

$a - 2 = 14$

$a = 14$ which is not possible [1]

[2, 6, 10] is the equivalence class of [2].

[TOPIC 2] Binary Operations

Summary

- **Binary Operation:** A binary operation * on a set A is a function * from $A \times A$ to A. We denote * (a, b) by $a * b$.
- An element $e \in X$ is the **identity element** for binary operation * : $X \times X \to X$, if $a * e = a = e * a \forall a \in X$.
- An element $a \in X$ is **invertible** for binary operation * : $X \times X \to X$, if there exists $b \in X$ such that $a * b = e = b *$ a where, e is the identity for the binary operation *. The element b is called inverse of a and is denoted by a^{-1}.
- An operation * on X is **commutative** if $a * b = b * a \forall a, b$ in X.
- An operation * on X is **associative** if $(a * b) * c = a * (b * c) \forall a, b, c$ in X.

PREVIOUS YEARS'

EXAMINATION QUESTIONS
TOPIC 2

1 Mark Questions

1. Let * be a binary operation, on the set of all non zero real numbers, given by $a * b = \frac{ab}{5}$ for all $a, b \in R - \{0\}$. Find the value of x, given that $2 * (x * 5) = 10$.

[DELHI 2014]

2. Let * be a 'binary operation on N given $a * b$ = LCM (a, b) for all $a, b \in N$. find 5*7 .

[ALL INDIA 2012]

3. Let * be a binary operation on N given by a * b = LCM (a, b) for all a,b ∈ N. Find 5*7.

[DELHI 2012]

4. If a*b denote the larger of 'a' and 'b' and if $a \bigcirc b = (a * b) + 3$, then write the value of $5 \bigcirc 10$, where * and O are binary operations.

[DELHI 2018]

5. Consider the binary operation *on the set {1, 2, 3, 4, 5} defined by a*b = min {a, b} write the operation table of the operation .

[DELHI 2011]

6 Marks Questions

6. Let A = Q × Q, where Q is the set of all rational numbers, and be a binary operation defined on A by $(a, b) * (c, d) = (ac, b + ad)$, for all $(a, b) (c, d) A$. Find

(i) the identity element in A.

(ii) the invertible element of A.

[ALL INDIA 2015]

7. Discuss the Commutativity and associativity of binary operation * defined on $A = Q - \{1\}$ by the rule $a * b = a - b + ab \; \forall a, b \in A$. Also find the identity element of * in A and hence find the invertible elements of A.

[DELHI 2017]

Solutions

1. Given that, $2 * (x * 5) = 10$

$$2 * \left(\frac{x5}{5}\right) = 10 \quad \left(\because a * b = \frac{ab}{5}\right)$$ [½]

$$\Rightarrow 2 * x = 10$$

$$\Rightarrow \frac{2x}{5} = 10$$

$$\therefore x = 10 \times \frac{5}{2} = 25$$ [½]

2. Given a*b = LCM (a, b)

5 * 7 = LCM (5, 7) = 35 [1]

3. 5 * 7 = LCM of 5 and 7 = 35 [1]

4. Given that $a \bigcirc b = (a * b) + 3$

$$5 \bigcirc 10 = (5 * 10) + 3$$
$$= 10 + 3 \quad [1]$$
$$= 13$$

5. Given : since * carries each pair (a, b) in R × R to unique element namely minimum of a and b lying in R,* is a binary operation

$a * b = \min(a, b)$

1 * 1 = 1

1 * 2 = 1

2 * 1 = 1

The operation table

$a * b$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

[1]

6. Let (x, y) be the identity element in Q × Q then

$(a, b) * (x, y) = (a, b) = (x, y) * (a, b) \forall (a, b) \in Q \times Q$

$\Rightarrow (ax, b + ay) = (a, b)$ [1]

⇒ a = ax and b = b + ay

⇒ x = 1 and y = 0

⇒ (1, 0) is the identity element in Q × Q [1]

Let (a, b) be the invertible element in Q × Q, then

There exists $(\alpha, \beta) \in Q \times Q$ such that

$(a, b) * (\alpha, \beta) = (\alpha, \beta) * (a, b) = (1, 0)$ [2]

$\Rightarrow (a\alpha, b + \alpha\beta) = (1, 0)$

$\Rightarrow \alpha = \frac{1}{a}, \beta = -\frac{b}{a}$ [1]

∴ The invertible element in A is $\left(\frac{1}{a}, -\frac{b}{a}\right)$ [1]

7. Commutativity

$a * b = a - b + ab, \ \forall a, b \in A$ [1]

$b * a = b - a + ba$

$b * a = -(a - b) + ab$

$a * b \neq b * a$

Thus, * is not commutative. [1]

Associativity

$(a * b) * c = (a - b + ab) * c, \quad \forall a, b, c \in A$

$= a - b + ab - c + (a - b + ab)c$

$= a - b + ab - c + ac - bc + ab$

$a * (b * c) = a * (b - c + bc), \quad \forall a, b, c \in A$ [1]

$= a - (b - c + bc) + a(b - c + bc)$

$= a - b + c - bc + ab - ac + abc$

$(a * b) * c \neq a * (b * c)$

Thus, * is not associative. [1]

Identity:

Let i be the identity.

Thus, $i * a = a * i$

Checking $i * a = a$

$i - a + ia = a$

$i(1 + a) = 2a$

$i = \frac{2a}{1 + a}, (1 + a) \neq 0$

Checking $a * i = a$ [1]

$a - i + ai = a$

$i(a - 1) = 0$

$i = \frac{0}{a - 1}, (a - 1) \neq 0$

Thus, i is not giving same value. [1]

Hence, No identity element exists.

Smart Notes ....

CHAPTER 2

Inverse Trigonometric Functions

Chapter Analysis of Previous Three Years' Board Exams

Number of Questions Asked in Exams

Topics	2018		2017		2016	
	Delhi	**All India**	**Delhi**	**All India**	**Delhi**	**All India**
Solution of Equations using inverse Trigonometric formula			4 marks	4 marks	4 marks	
Properties of Inverse Trigonometric Functions	1, 2 marks	1, 2 marks				4 marks

Inverse Trigonometric Functions

Summary

Definition of inverse trigonometric functions:

Inverse trigonometric functions are the inverse of trigonometric functions, we can represents them by using "-1" or arc on trigonometric functions. Also the Range of trigonometric function becomes the Domain of Inverse trigonometric function.

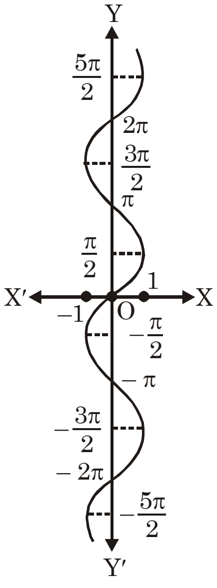
For ex: $x = \sin y$ will be represented as $y = \arcsin x$ or $y = \sin^{-1} x$.

Range of $x = \sin y$ is [–1, 1] and Domain of $y = \arcsin x$ is [–1, 1].

- The inverse trigonometric functions are also called as **Inverse Circular Functions**.
- **Function:** $y = \sin^{-1}x$

  Domain: [–1, 1]

  Range: $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

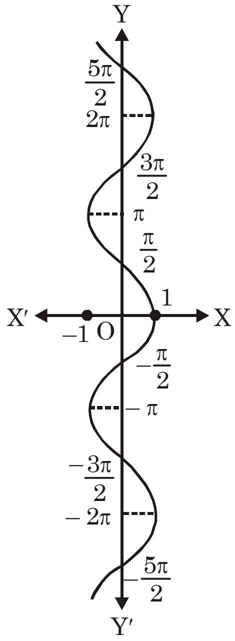


$y = \sin^{-1} x$

- **Function:** $y = \cos^{-1}x$

  Domain: [–1, 1]

  Range: $[0, \pi]$

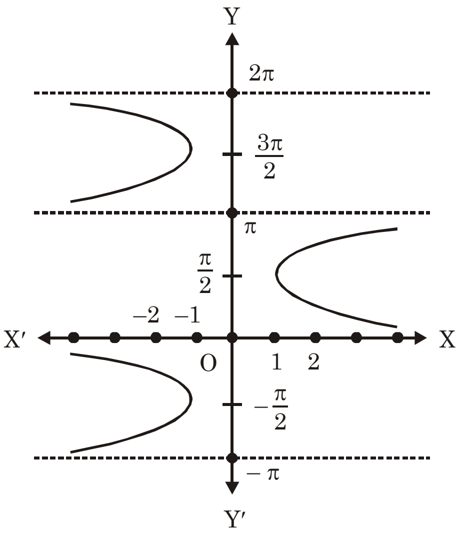


$y = \cos^{-1} x$

- **Function:** $y = \text{cosec}^{-1}x$

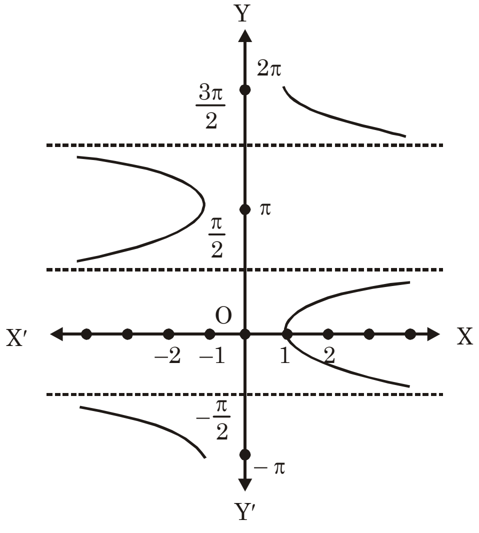
  Domain: R – (–1, 1)

  Range: $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$



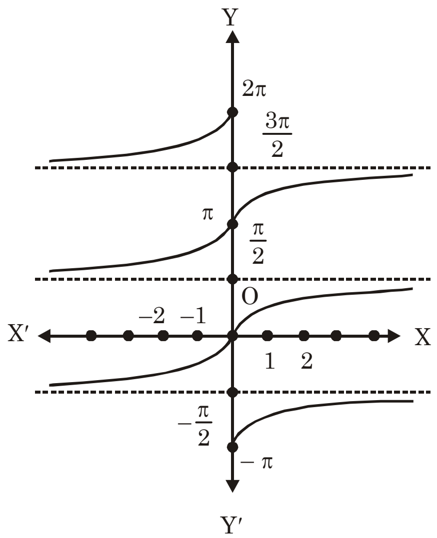
$y = \text{cosec}^{-1} x$

- **Function:** $y = \sec^{-1}x$
  Domain: R – (–1, 1)
  Range: $[0, \pi] - \left\{\frac{\pi}{2}\right\}$



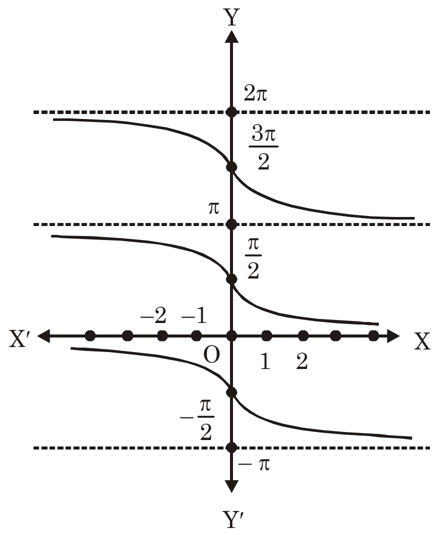
$y = \sec^{-1} x$

- **Function:** $y = \tan^{-1}x$
  Domain: R
  Range: $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$



$y = \tan^{-1} x$

- **Function:** $y = \cot^{-1}x$
  Domain: R
  Range: (0, π)



$y = \cot^{-1} x$

- **Properties:**
  - $\sin^{-1}(\sin x) = x$
  - $\cos^{-1}(\cos x) = x$
  - $\tan^{-1}(\tan x) = x$
  - $\text{cosec}^{-1}(\text{cosec } x) = x$
  - $\sec^{-1}(\sec x) = x$
  - $\cot^{-1}(\cot x) = x$
  - $\sin^{-1}(x) = \text{cosec}^{-1}\left(\frac{1}{x}\right),\ x \in [-1, 1]$
  - $\text{cosec}^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right),\ x \in (-\infty, -1] \cup [1, \infty)$
  - $\cos^{-1}(x) = \sec^{-1}\left(\frac{1}{x}\right),\ x \in [-1, 1]$
  - $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right),\ x \in (-\infty, -1] \cup [1, \infty)$
  - $\tan^{-1}(x) = \begin{cases} \cot^{-1}\left(\frac{1}{x}\right), & x > 0 \\ -\pi + \cot^{-1}\left(\frac{1}{x}\right), & x < 0 \end{cases}$
  - $\cot^{-1}(x) = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right), & x > 0 \\ \pi + \tan^{-1}\left(\frac{1}{x}\right), & x < 0 \end{cases}$

- $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2},\ x \in [-1,1]$
- $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2},\ x \in R$
- $\text{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2},\ |x| \geq 1$
- $\sin^{-1}(-x) = -\sin^{-1} x$
- $\cos^{-1}(-x) = \pi - \cos^{-1} x$
- $\tan^{-1}(-x) = -\tan^{-1} x$
- $\cot^{-1}(-x) = \pi - \cot^{-1} x$
- $\sec^{-1}(-x) = \pi - \sec^{-1} x$
- $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$
- $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$
- $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}$
- $2\tan^{-1} x = \begin{cases} \sin^{-1}\left(\frac{2x}{1+x^2}\right), & |x| \leq 1 \\ \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), & |x| \geq 0 \\ \tan^{-1}\left(\frac{2x}{1-x^2}\right), & -1 < x < 1 \end{cases}$

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

1 Mark Questions

1. Write the principal value of

$\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right)$.

[ALL INDIA 2013]

2. Write the value of $\tan\left(2\tan^{-1}\frac{1}{5}\right)$.

[DELHI 2013]

3. Write the principal value of

$\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(\frac{1}{2}\right)$.

[DELHI 2012]

4. Write the value of $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right]$

[DELHI 2011]

5. Write the principal value of $\tan^{-1}(-1)$

[ALL INDIA 2011]

6. Write the principal value of

$\tan^{-1}\left(\sqrt{3}\right) - \cot^{-1}\left(-\sqrt{3}\right)$.

[ALL INDIA 2013, 2018]

7. Write the value of $\tan^{-1}\left[2\sin\left[2\cos^{-1}\frac{\sqrt{3}}{2}\right]\right]$.

[All India 2013]

8. If $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}, xy < 1,$ then write the value of $x + y + xy$.

[ALL INDIA 2014]

4 Marks Questions

9. Find the value of the following:

$\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right], |x| < 1, y > 0$

and $xy < 1$.

[DELHI 2013]

10. If $\sin\left[\cot^{-1}(x+1)\right] = \cos\left(\tan^{-1} x\right)$, then find x.

[DELHI 2015]

11. Prove that

$\tan\left\{\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} + \tan\left\{\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} = \frac{2b}{a}$.

[DELHI 2017]

12. Prove that:

$$\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$$

[DELHI 2016]

13. Solve for : $2\tan^{-1}(\cos x)=\tan^{-1}(2\text{cosec } x)$

[DELHI 2016]

14. Prove that:

$$2\tan^{-1}\left(\frac{1}{5}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)+2\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$$

[DELHI 2014]

15. Prove that :

$$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)=\frac{\pi}{4}-\frac{x}{2}, x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right).$$

[ALL INDIA 2012]

16. Prove the :

$$\sin^{-1}\left(\frac{8}{17}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\cos^{-1}\left(\frac{36}{85}\right).$$

[DELHI 2012]

17. Find the value of $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right).$

[DELHI 2011]

18. Prove the following :

$$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]=\frac{x}{2}, \text{x}\in\left(0,\frac{\pi}{4}\right).$$

[DELHI 2011]

19. Prove that: $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

[DELHI 2013]

20. Prove the following:

$$2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\left(\frac{31}{17}\right)$$

[ALL INDIA 2011]

21. Solve the following equation for x:

$$\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x, x>0$$

[DELHI 2011]

22. Show that : $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3}$

[ALL INDIA 2013]

23. Solve the following equations:

$$\cos\left(\tan^{-1}x\right)=\sin\left(\cot^{-1}\frac{3}{4}\right)$$

[ALL INDIA 2013]

24. Prove that

$$\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x, \frac{-1}{\sqrt{2}}\le x\le 1$$

[ALL INDIA 2014]

25. If $\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4}$, find the value of x.

[DELHI 2014]

26. Prove that:

$$2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)=\cos^{-1}\left(\frac{a\cos x+b}{a+b\cos x}\right)$$

[ALL INDIA 2015]

27. Solve the following for x:

$$\tan^{-1}\left(\frac{x-2}{x-3}\right)+\tan^{-1}\left(\frac{x+2}{x+3}\right)=\frac{\pi}{4}, |x|<1.$$

[ALL INDIA 2015]

28. Solve for x:

$$\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$$

[ALL INDIA 2015]

29. Prove that:

$$\tan^{-1}\left(\frac{6x-8x^3}{1-12x^2}\right)-\tan^{-1}\left(\frac{4x}{1-4x^2}\right)=\tan^{-1}2x; |2x|<\frac{1}{\sqrt{3}}$$

[ALL INDIA 2016]

30. If $\tan^{-1}\frac{x-3}{x-4}+\tan^{-1}\frac{x+3}{x+4}=\frac{\pi}{4}$, then find the value of x

[ALL INDIA 2017]

Solutions

1. $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)$.

$\left(\cos^{-1}(-\theta)=\pi-\cos^{-1}\theta\right)$

$$=\frac{\pi}{4}+\pi-\cos^{-1}\left(\frac{1}{2}\right)$$ [½]

$$=\frac{\pi}{4}+\pi-\frac{\pi}{3}$$

$$=\frac{3\pi+12\pi-4\pi}{12}=\frac{11\pi}{12}.$$ [½]

2. Given : $\tan\left(2\tan^{-1}\frac{1}{5}\right)$

$\left[2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$

$$=\tan\left[\tan^{-1}\left(\frac{2\left(\frac{1}{5}\right)}{1-\left(\frac{1}{5}\right)^2}\right)\right]$$ [½]

$$=\frac{\frac{2}{5}}{1-\frac{1}{25}}$$

$$=\frac{\frac{2}{5}}{\frac{25-1}{25}}=\frac{2}{5}\times\frac{25}{24}=\frac{5}{12}.$$ [½]

3. Given : $\cos^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\left(\frac{1}{2}\right)$.

$\left[\sin(-\theta)=-\sin\theta\right]$

$$=\cos^{-1}\left(\cos\frac{\pi}{3}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$$ [½]

$$=\frac{\pi}{3}+2\sin^{-1}\left(\sin\frac{\pi}{6}\right)$$

$$=\frac{\pi}{3}+2\left(\frac{\pi}{6}\right)=\frac{\pi}{3}+\frac{\pi}{3}=\frac{2\pi}{3}$$ [½]

4. Given : $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right]$

Using $\sin^{-1}(-x)=-\sin^{-1}x$

$$=\sin\left[\frac{\pi}{3}+\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$=\sin\left[\frac{\pi}{3}+\frac{\pi}{6}\right]=\sin\left[\frac{\pi}{2}\right]=1$$ [1]

5. Let $\tan^{-1}(-1)=y$

$\tan y=-1$

$\tan y=-\tan\left(\frac{\pi}{4}\right)$ [1]

$\tan y=\tan\left(-\frac{\pi}{4}\right)$

$y=-\frac{\pi}{4}$

6. $\tan^{-1}\left(\sqrt{3}\right)=\frac{\pi}{3}$

$\cot^{-1}\left(-\sqrt{3}\right)=\pi-\frac{\pi}{6}$

Hence

$$\frac{\pi}{3}-\left(\pi-\frac{\pi}{6}\right)=-\frac{\pi}{2}$$ [1]

7. $\because\cos^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{6}$

$$\tan^{-1}\left[2\sin\left[2\cos^{-1}\frac{\sqrt{3}}{2}\right]\right]$$

$$=\tan^{-1}\left(2\sin\left(2.\frac{\pi}{6}\right)\right)$$ [½]

$$=\tan^{-1}\left(2\sin\frac{\pi}{3}\right)$$

$$=\tan^{-1}\left[2\frac{\sqrt{3}}{2}\right]$$

$$=\tan^{-1}\sqrt{3}=\frac{\pi}{3}$$ [½]

8. $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$

$\Rightarrow \tan^{-1}\frac{x+y}{1-xy}=\frac{\pi}{4}$

$\Rightarrow \frac{x+y}{1-xy}=\tan\frac{\pi}{4}$

$\Rightarrow \frac{x+y}{1-xy}=1 \text{ or}, x+y=1-xy$ [1]

Or, $x+y+xy=1$

9. Given:

$\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1-y^2}\right]$

$[Let\ x=\tan A, y=\tan B]$

$=\tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2\tan x}{1+\tan^2 x}\right)+\cos^{-1}\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)\right]$

[1]

$=\tan\frac{1}{2}\left[\sin^{-1}(\sin 2x)+\cos^{-1}(\cos 2x)\right]$ [2]

$=\tan(x+y)$

$=\frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{x+y}{1-xy}$ [1]

10. $\sin\left[\cot^{-1}(x+1)\right]=\cos\left(\tan^{-1}x\right),$

Let $\tan^{-1}x=B$

$\Rightarrow \tan B=\frac{x}{1}$

$\therefore \cos B=\frac{1}{\sqrt{1+x^2}}$ [1]

Let $\cot^{-1}(x+1)=A$

$\therefore\ \frac{x+1}{1}=\cot A$

$\therefore\ sin\ A=\frac{1}{\sqrt{(x+1)^2+1}}$ [1]

Now sin A = cos B

$\therefore \frac{1}{\sqrt{(x+1)^2+1}}=\frac{1}{\sqrt{1+x^2}}$ [1]

$\therefore \sqrt{(x+1)^2+1}=\sqrt{1+x^2}$

Now squaring on both sides, we get

$\therefore (x+1)^2+1=1+x^2$

$\therefore\ x^2+2x+1=x^2$

$\therefore 2x=-1$

$\Rightarrow\ x=-\frac{1}{2}$ [1]

11. Given :

$\tan\left\{\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right\}+\tan\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right\}=\frac{2b}{a}.$

Proceeding from left hand side

$\tan\left\{\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right\}+\tan\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right\}$

Let $\frac{1}{2}\cos^{-1}\frac{a}{b}=c$ [1]

$=\tan\left\{\frac{\pi}{4}+c\right\}+\tan\left\{\frac{\pi}{4}-c\right\}$

Now $\tan\left(\frac{\pi}{4}+c\right)=\frac{1+\tan c}{1-\tan c}$ and

$\tan\left(\frac{\pi}{4}-c\right)=\frac{1-\tan c}{1+\tan c}$

$=\frac{1+\tan c}{1-\tan c}+\frac{1-\tan c}{1+\tan c}$

$=\frac{(1+\tan c)^2+(1-\tan c)^2}{(1-\tan c)(1+\tan c)}$ [1]

$=\frac{1+\tan^2 c+2\tan c+1+\tan^2 c-2\tan c}{1-\tan^2 c}$

$=\frac{2+2\tan^2 c}{1-\tan^2 c}$

$= \dfrac{2\left(1+\tan^2 c\right)}{1-\tan^2 c}$

$= \dfrac{2\left(1+\dfrac{\sin^2 c}{\cos^2 c}\right)}{1-\dfrac{\sin^2 c}{\cos^2 c}}$

$= \dfrac{2\left(\dfrac{\cos^2 c+\sin^2 c}{\cos^2 c}\right)}{\dfrac{\cos^2 c-\sin^2 c}{\cos^2 c}}$

Using $\cos^2 c-\sin^2 c=\cos 2c$

$= \dfrac{2(1)}{\cos 2c}$ [1]

As $\cos^{-1}\dfrac{a}{b}=2c$

$= \dfrac{2(1)}{\cos\left(\cos^{-1}\dfrac{a}{b}\right)}$

$= \dfrac{2}{\dfrac{a}{b}} = \dfrac{2b}{a}$ [1]

L.H.S = R.H.S. Hence, proved.

12. L.H.S = $\tan^{-1}\dfrac{1}{5}+\tan^{-1}\dfrac{1}{7}+\tan^{-1}\dfrac{1}{3}+\tan^{-1}\dfrac{1}{8}$

$=\left[\tan^{-1}\left(\dfrac{1}{3}\right)+\tan^{-1}\left(\dfrac{1}{5}\right)\right]+\left[\tan^{-1}\left(\dfrac{1}{7}\right)+\tan^{-1}\left(\dfrac{1}{8}\right)\right]$

$=\tan^{-1}\left(\dfrac{\dfrac{1}{3}+\dfrac{1}{5}}{1-\dfrac{1}{3}\times\dfrac{1}{5}}\right)+\tan^{-1}\left(\dfrac{\dfrac{1}{7}+\dfrac{1}{8}}{1-\dfrac{1}{7}\times\dfrac{1}{8}}\right)$ [1]

$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\dfrac{x+y}{1-xy}\right) \text{ if } xy<1\right]$

$=\tan^{-1}\left(\dfrac{5+3}{15-1}\right)+\tan^{-1}\left(\dfrac{8+7}{56-1}\right)$

$=\tan^{-1}\left(\dfrac{8}{14}\right)+\tan^{-1}\left(\dfrac{15}{55}\right)$

$=\tan^{-1}\left(\dfrac{4}{7}\right)+\tan^{-1}\left(\dfrac{3}{11}\right)$ [1]

$=\tan^{-1}\left(\dfrac{\dfrac{4}{7}+\dfrac{3}{11}}{1-\dfrac{4}{7}\times\dfrac{3}{11}}\right)=\tan^{-1}\left(\dfrac{\dfrac{44+21}{77}}{\dfrac{77-12}{77}}\right)$ [1]

$=\tan^{-1}\left(\dfrac{44+21}{77-12}\right)=\tan^{-1}\left(\dfrac{65}{65}\right)$

$=\tan^{-1}(1)=\dfrac{\pi}{4}=$ R.H.S [1]

13. $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$

$\therefore \tan^{-1}\left(\dfrac{2\cos x}{1-\cos^2 x}\right)=\tan^{-1}\left(\dfrac{2}{\sin x}\right)$ [1]

$\left[\because 2\tan^{-1}(A)=\tan^{-1}\left(\dfrac{2A}{1-A^2}\right)\right]$ [1]

$\Rightarrow \dfrac{2\cos x}{\sin^2 x}=\dfrac{2}{\sin x}$ $\quad\Rightarrow \dfrac{\cos x}{\sin x}=1$ [1]

$\Rightarrow \cot x=1$ $\quad\Rightarrow \cot x=\cot\dfrac{\pi}{4}$

$\Rightarrow x=\dfrac{\pi}{4}$ [1]

14. $LHS=2\tan^{-1}\left(\dfrac{1}{5}\right)+\sec^{-1}\left(\dfrac{5\sqrt{2}}{7}\right)+2\tan^{-1}\left(\dfrac{1}{8}\right)$

Let $A=Sec^{-1}\left(\dfrac{5\sqrt{2}}{7}\right)$

$=2\tan^{-1}\left(\dfrac{1}{5}\right)+2\tan^{-1}\left(\dfrac{1}{8}\right)+A$

Now, $Sec\,A=\left(\dfrac{5\sqrt{2}}{7}\right)$

$\therefore \tan A=\dfrac{1}{7}\Rightarrow A=\tan^{-1}\left(\dfrac{1}{7}\right)$

$=2\left[\tan^{-1}\left(\dfrac{1}{5}\right)+\tan^{-1}\left(\dfrac{1}{8}\right)\right]+\tan^{-1}\left(\dfrac{1}{7}\right)$

Using identity

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$ [1]

$$=2\tan^{-1}\left(\frac{8+5}{40-1}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$=2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$ [1]

Using identity $2\tan^{-1}A=\tan^{-1}\left(\frac{2A}{1-A^2}\right)$

$$=\tan^{-1}\left(\frac{2\left(\frac{1}{3}\right)}{1-\frac{1}{9}}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$=\tan^{-1}\left(\frac{2}{3}\times\frac{9}{8}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$=\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

Again, using identity

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$ [1]

$$=\tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}.\frac{1}{7}}\right)$$

$$=\tan^{-1}\left(\frac{21+4}{28-3}\right)$$

$$=\tan^{-1}\left(\frac{25}{25}\right)=\tan^{-1}(1)=\frac{\pi}{4}=R.H.S$$ [1]

Hence, proved.

15. $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

$$\left[\sin A=2\sin\frac{A}{2}\cos\frac{A}{2},\quad 1+\cos A=2\cos^2\frac{A}{2}\right]$$

$$=\tan^{-1}\left(\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}\right)$$ [1]

$$\tan^{-1}\left(\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right)$$ [2]

$$=\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)=\frac{\pi}{4}-\frac{x}{2}=R.H.S.$$ [1]

16. Given: $\sin^{-1}\left(\frac{8}{17}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\cos^{-1}\left(\frac{36}{85}\right)$.

Let $A=\frac{3}{5}$

$$\sin A=\frac{3}{5}$$

$$\cos^2A=1-\sin^2A$$

$$=1-\left(\frac{3}{5}\right)^2$$

$$=\frac{25-9}{25}=\frac{16}{25}$$ [1]

$$\therefore \cos A=\sqrt{\frac{16}{25}}=\frac{4}{5}\qquad \text{(i)}$$

Let $B=\sin^{-1}\frac{8}{17}$

$$\sin B=\frac{8}{17}$$ [1]

$$\cos^2B=1-\sin^2B$$

$$=1-\left(\frac{8}{17}\right)^2$$

$$=\frac{289-64}{289}$$

$$=\frac{225}{289}$$

$$\cos B = \sqrt{\frac{225}{289}} = \frac{15}{17} \quad ...(ii) \quad [1]$$

Now $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$= \frac{4}{5}\times\frac{15}{17} - \frac{3}{5}\times\frac{8}{17}$$

$$= \frac{60}{85} - \frac{24}{85} = \frac{36}{85}$$

$$\Rightarrow A + B = \cos^{-1}\left(\frac{36}{85}\right) \quad ...(iii)$$

From (i) ,(ii) and (iii), we get

$$\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{36}{85} \quad [1]$$

17. Given :

$$\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$$

$$= \tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{\frac{x}{y}-1}{\frac{x}{y}+1}\right) \quad [1½]$$

$$= \tan^{-1}\left(\frac{x}{y}\right) - \left[\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}(1)\right] \quad [1½]$$

$$\left[\because \tan^{-1}\left(\frac{A-B}{1+AB}\right) = \tan^{-1} A - \tan^{-1} B\right]$$

$$= \tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\tan\frac{\pi}{4}\right)$$

$$= \frac{\pi}{4} \quad [1]$$

18. Given : $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

$$\left[\begin{array}{l} 1+\cos x = 2\cos^2\frac{x}{2} \\ \textit{and} \ \ \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \end{array}\right]$$

On conjugating we get

$$\cot^{-1}\left(\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)\times\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right)\right)$$

[½]

$$= \cot^{-1}\left(\frac{1+\sin x+1-\sin x+2\sqrt{1+\sin x}\sqrt{1-\sin x}}{1+\sin x-(1-\sin x)}\right)$$

$$= \cot^{-1}\left(\frac{2+2\cos x}{2\sin x}\right) = \cot^{-1}\left(\frac{2(1+\cos x)}{2\sin x}\right) \quad [1½]$$

$$= \cot^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$$= \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2} = R.H.S \quad [2]$$

19. Given $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

Using $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$\tan^{-1}\left[\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\times\frac{1}{5}}\right] + \tan^{-1}\frac{1}{8} = \frac{\pi}{4} \quad [1]$$

$$\tan^{-1}\left[\frac{\frac{7}{10}}{\frac{9}{10}}\right] + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4} \quad [1]$$

$$\tan^{-1}\left[\frac{7}{9}\right] + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4} \quad [1]$$

Again using $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$= \tan^{-1}\left[\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\times\frac{1}{8}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{56+9}{72}}{\frac{72-7}{72}}\right]$$

$$= \tan^{-1}\left[\frac{65}{65}\right]$$

$$= \frac{\pi}{4}$$ [1]

Proved.

20. As we know that, $2\tan^{-1}x = \tan^{-1}\left[\frac{2x}{1-x^2}\right]$ [1]

L.H.S

$$2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left[\frac{2.\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right] + \tan^{-1}\frac{1}{7}$$ [1]

$$= \tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

Also, $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$= \tan^{-1}\left[\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}.\frac{1}{7}}\right]$$ [1]

$$= \tan^{-1}\left(\frac{31}{17}\right)$$ [1]

= R.H.S

Hence, proved.

21. $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$

$$\therefore \tan^{-1}\left(\frac{1-x}{1+1\times x}\right) = \frac{1}{2}\tan^{-1}x$$

$$\therefore \tan^{-1}(1) - \tan^{-1}x = \frac{1}{2}\tan^{-1}x$$ [1½]

$$\therefore \tan^{-1}(1) = \frac{3}{2}\tan^{-1}x$$

$$\therefore \frac{\pi}{4} = \frac{3}{2}\tan^{-1}x$$ [1½]

$$\therefore \tan^{-1}x = \frac{\pi}{6}$$

$$\therefore x = \tan\frac{\pi}{6}$$

$$\therefore x = \frac{1}{\sqrt{3}}$$ [1]

22. Let $\frac{1}{2}\sin^{-1}\frac{3}{4} = \theta$ then $\frac{3}{4} = \sin 2\theta$ [1]

Now $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \tan\theta$

If $\sin 2\theta = \frac{3}{4}$ then

$$\frac{2\tan\theta}{1+\tan^2\theta} = \frac{3}{4}$$ [1]

$$\therefore 8\tan\theta = 3 + 3\tan^2\theta$$

$$\therefore 3\tan^2\theta - 8\tan\theta + 3 = 0$$

$$\therefore \tan\theta = \frac{8 \pm \sqrt{64 - 4\times 3\times 3}}{6}$$ [1]

$$\therefore \tan\theta = \frac{8\pm\sqrt{28}}{6} = \frac{4\pm\sqrt{7}}{3}$$

$$\therefore \tan\theta = \frac{4+\sqrt{7}}{3} \text{ or } \frac{4-\sqrt{7}}{3}$$

$$\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$$ [1]

Hence proved.

23. $\cos(\tan^{-1} x)$

Let $\tan^{-1} x = \theta$

$\Rightarrow x = \tan\theta$ [1]

$\cos\theta = \dfrac{1}{\sqrt{1+\tan^2\theta}} = \dfrac{1}{\sqrt{1+x^2}}$ [1]

Hence $\cos\left(\tan^{-1} x\right) = \dfrac{1}{\sqrt{1+x^2}}$

RHS

Let $\cot^{-1}\dfrac{3}{4} = \theta$

$\Rightarrow \dfrac{3}{4} = \cot\theta$

Then $\sin\theta = \dfrac{1}{\sqrt{1+\cot^2\theta}} = \dfrac{1}{\sqrt{1+\dfrac{9}{16}}} = \dfrac{4}{5}$ [1]

Now L.H.S = R.H.S

$\therefore \dfrac{1}{\sqrt{1+x^2}} = \dfrac{4}{5}$

$\therefore 25 = 16 + 16x^2$

$\therefore x^2 = \dfrac{9}{16}$

$\therefore x = \dfrac{3}{4}$ [1]

24. $\tan^{-1}\left[\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right] = \dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1} x, \dfrac{-1}{\sqrt{2}} \le x \le 1$

L.H.S.

Let $x = \cos 2\theta$ [1]

$\therefore \tan^{-1}\left[\dfrac{\sqrt{1+\cos2\theta}-\sqrt{1-\cos2\theta}}{\sqrt{1+\cos2\theta}+\sqrt{1-\cos2\theta}}\right]$

$= \tan^{-1}\left[\dfrac{\sqrt{1+2\cos^2\theta-1}-\sqrt{1-1+2\sin^2\theta}}{\sqrt{1+2\cos^2\theta-1}+\sqrt{1-1+2\sin^2\theta}}\right]$ [1]

$= \tan^{-1}\left[\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right]$

$= \tan^{-1}\left[\dfrac{1-\tan\theta}{1+\tan\theta}\right]$

$= \tan^{-1}\left[\dfrac{\tan\left(\dfrac{\pi}{4}\right)-\tan\theta}{1+\tan\left(\dfrac{\pi}{4}\right).\tan\theta}\right]$ [1]

$= \tan^{-1}\left[\tan\left(\dfrac{\pi}{4}-\theta\right)\right]$

$= \dfrac{\pi}{4} - \theta$

$= \dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1} x$ [1]

= RHS

Hence, proved.

25. $\tan^{-1}\left(\dfrac{x-2}{x-4}\right) + \tan^{-1}\left(\dfrac{x+2}{x+4}\right) = \dfrac{\pi}{4}$

Now using formula,

$\tan^{-1}(a) + \tan^{-1}(b) = \tan^{-1}\left(\dfrac{a+b}{1-ab}\right)$

$\tan^{-1}\left[\dfrac{\dfrac{x-2}{x-4}+\dfrac{x+2}{x+4}}{1-\left(\dfrac{x-2}{x-4}\right)\left(\dfrac{x+2}{x+4}\right)}\right] = \dfrac{\pi}{4}$ [1]

$\Rightarrow \tan^{-1}\left[\dfrac{(x-2)(x+4)+(x+2)(x-4)}{(x-4)(x+4)-(x-2)(x+2)}\right] = \dfrac{\pi}{4}$ [1]

$\therefore \dfrac{(x-2)(x+4)+(x+2)(x-4)}{(x-4)(x+4)-(x-2)(x+2)} = 1$

$\therefore \dfrac{x^2-8+2x+x^2-8-2x}{x^2-16-x^2+4} = 1$

$\therefore \dfrac{2x^2-16}{-12} = 1$ [1]

$\therefore 2x^2 = -12 + 16 = 4$

$\therefore x^2 = 2$

$\therefore x = \pm\sqrt{2}$ [1]

26. $\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$

$$= \cos^{-1}\left(\frac{1-\frac{a-b}{a+b}\tan^2\frac{x}{2}}{1+\frac{a-b}{a+b}\tan^2\frac{x}{2}}\right) \quad [1]$$

$$= \cos^{-1}\left(\frac{a+b-a\tan^2\frac{x}{2}+btan^2\frac{x}{2}}{a+b+a\tan^2\frac{x}{2}-btan^2\frac{x}{2}}\right)$$

$$= \cos^{-1}\left\{\frac{a\left(1-\tan^2\frac{x}{2}\right)+b\left(1+\tan^2\frac{x}{2}\right)}{a\left(1+\tan^2\frac{x}{2}\right)+b\left(1-\tan^2\frac{x}{2}\right)}\right\} \quad [1]$$

$$= \cos^{-1}\left\{\frac{a\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}+b}{a+b\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}\right\} \quad [1]$$

$$= \cos^{-1}\left\{\frac{acosx+b}{a+bcosx}\right\} \quad [1]$$

27. $\tan^{-1}\left(\frac{x-2}{x-3}\right)+\tan^{-1}\left(\frac{x+2}{x+3}\right) = \frac{\pi}{4}$,

Using $\tan^{-1}(a)+\tan^{-1}(b) = \tan^{-1}\left(\frac{a+b}{1-ab}\right)$

$$\therefore \tan^{-1}\left(\frac{\frac{x-2}{x-3}+\frac{x+2}{x+3}}{1-\frac{x-2}{x-3}.\frac{x+2}{x+3}}\right) = \frac{\pi}{4} \quad [1½]$$

$$\therefore \tan^{-1}\left(\frac{2x^2-12}{-5}\right) = \frac{\pi}{4} \quad [1]$$

$$\therefore \frac{2x^2-12}{-5} = 1$$

$$\therefore x^2 = \frac{7}{2} \quad [1]$$

$$\therefore x = \sqrt{\frac{7}{2}} \quad [½]$$

28. $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1) = \tan^{-1}3x$

$\therefore \tan^{-1}(x-1)+\tan^{-1}(x+1) = \tan^{-1}3x-\tan^{-1}x$ [½]

Now, $\tan^{-1}(a)+\tan^{-1}(b) = \tan^{-1}\left(\frac{a+b}{1-ab}\right)$

$$\therefore \tan^{-1}\left(\frac{x-1+x+1}{1-(x-1)(x+1)}\right) = \tan^{-1}\left(\frac{3x-x}{1+3x^2}\right) \quad [1]$$

$$\therefore \frac{2x}{1-\left(x^2-1\right)} = \frac{2x}{1+3x^2}$$

$$\therefore 2x\left(1+3x^2\right) = 2x\left(2-x^2\right)$$

$$\therefore 2x\left[\left(1+3x^2\right)-2+x^2\right] = 0 \quad [1½]$$

$$\therefore x\left(4x^2-1\right) = 0$$

$$\therefore x = 0; x^2 = \frac{1}{4}$$

$$\therefore x = \pm\frac{1}{2}$$

$$\therefore x = 0, +\frac{1}{2}, -\frac{1}{2} \quad [1]$$

29. $\tan^{-1}\left(\frac{3(2x)-(2x)^3}{1-3(2x)^2}\right)-\tan^{-1}\left(\frac{2(2x)}{1-(2x)^2}\right)$

Let $2x=\tan\theta$ $\quad |2x|\le\frac{1}{\sqrt{3}}$ [1]

$=\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)-\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$

$=\tan^{-1}(\tan 3\theta)-\tan^{-1}(\tan 2\theta)$ [2]

$=3\theta-2\theta=\theta=\tan^{-1}2x$ [1]

30. $\tan^{-1}\left[\frac{\frac{x-3}{x-4}+\frac{x+3}{x+4}}{1-\left(\frac{x^2-9}{x^2-16}\right)}\right]=\frac{\pi}{4}$

$\therefore \frac{(x+4)(x-3)+(x+3)(x-4)}{(x^2-16)-(x^2-9)}=1$ [2]

$\therefore 2x^2-24=-7$

$\therefore 2x^2=-7+24$

$\therefore x^2=\frac{17}{2}$ [1]

$\therefore x=\pm\sqrt{\frac{17}{2}}$ [1]

Smart Notes ....

Smart Notes ....

CHAPTER 3

Matrices

Chapter Analysis of Previous Three Years' Board Exams

Number of Questions Asked in Exams

Topics	2018		2017		2016	
	Delhi	**All India**	**Delhi**	**All India**	**Delhi**	**All India**
Operation on Matrix				6 marks		1 marks
Symmetric Matrix					1 marks	
Inverse of Matrix	2, 6 marks	2, 6 marks	1 mark		6 marks	
Properties of Matrix					1 marks	
Solution of linear Equation			6 marks		4 marks	4 marks
Order of Matrix				1 marks		1 marks
Skew Symmetric Matrix	1 marks	1 marks	2 marks	2 marks		

[TOPIC 1] Matrix and Operations on Matrices

Summary

- A **matrix** is an ordered rectangular array of numbers or functions. The numbers are called the **elements** of the matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & \dots & a_{3n} \\ . & . & . & \dots & \dots & . \\ . & . & . & \dots & \dots & . \\ a_{m1} & a_{m2} & a_{m3} & \dots & \dots & a_{mn} \end{bmatrix}_{m\times n}$$

Example: $A = \begin{bmatrix} 4 & \frac{1}{2} & -2 \\ 0 & 3 & 5 \\ 1 & \sqrt{6} & -7 \end{bmatrix}$

- The order of the matrix is determined by $m \times n$ where m is the number of rows and n is the number of columns.
- The matrix with $m \times n$ order can be represented as $A = \left[a_{ij}\right]_{m\times n}$; $i, j \notin \mathbb{N}$ also $1 \le i \le m, 1 \le j \le n,$.

Types of Matrices

- **Column matrix** is a matrix which has only 1 column. It is defined as $A = [a_{ij}]_{m\times 1}$.

Example: $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}_{3\times 1}$

- **Row matrix** is a matrix which has only row. It is defined as $B = [b_{ij}]_{1\times n}$.

Example: $\begin{bmatrix} \sqrt{2} & -1 & 3 \end{bmatrix}_{1\times 3}$

- **The square matrix** is the matrix which has equal number of rows and columns i.e. the matrix in which $m = n$. It is defined as $A = [a_{ij}]_{m\times m}$.

Example: $\begin{bmatrix} 3 & 9 & 1 \\ 7 & 6 & 3 \\ 9 & 0 & 2 \end{bmatrix}_{3\times 3}$

The square matrix $A = [a_{ij}]_{m\times m}$ is said to be a diagonal matrix if all its non-diagonal elements are zero. It is defined as $A = [a_{ij}]_{m\times m}$ if $a_{ij} = 0$, when $i \ne j$.

- A **scalar matrix** is the one in which the diagonal elements of a diagonal matrix are equal. It is defined as $A = [a_{ij}]_{m\times m}$ if $a_{ij} = 0$, when $i \ne j$ and $a_{ij} = k$, when $i = j$, where k is some constant.

Example: $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

- **Identity matrix** is the square matrix where the diagonal elements are all 1 and rest are all zero. It is defined as $A = [a_{ij}]_{m\times m}$ where $a_{ij} = 1$ if $i = j$ and $a_{ij} = 0$ if $i \ne j$.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

- **A zero matrix** is the one in which all the elements are zero.

Example: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$

- Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if they are of the same order and also each element of matrix A is equal to the corresponding element of Matrix B.
- The **sum of the two matrices** $A = [a_{ij}]$ and $B = [b_{ij}]$ of same order $m \times n$ is defined as $C = [c_{ij}]_{m\times n}$ where $c_{ij} = a_{ij} + b_{ij}$.
- If x is a scalar and $A = [a_{ij}]_{m\times n}$ is a matrix, then xA is the matrix obtained by multiplying each element of the matrix by the scalar x. It can be defined as $xA = x[a_{ij}]_{m\times m} = [x(a_{ij})]_{m\times n}$.
- $-A$ denotes the negative of a matrix. $-A = (-1)A$.
- The **difference of the two matrices** $A = [a_{ij}]$ and $B = [b_{ij}]$ of same order $m \times n$ is defined as $C = [c_{ij}]_{m\times n}$ where $c_{ij} = a_{ij} - b_{ij}$.
- If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same order, say $m \times n$ then $A + B = B + A$. It is called the **commutative law**.

- If $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ are the three matrices of same order, say $m \times n$, then $(A + B) + C = A + (B + C)$. It is called the **associative law**.
- If $B = [b_{ij}]$ is a matrix of order $m \times n$ and O is a zero matrix of the order $m \times n$, then $B + O = O + B = B$. O is the additive identity for matrix addition.
- If $B = [b_{ij}]$ is a matrix of order $m \times n$ then we have another matrix as $-B = [-b_{ij}]$ of the order $m \times n$ such that $B + (-B) = (-B) + B = O$. So, $-B$ is the additive inverse of B.
- If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same order, say $m \times n$, and x and y are the scalars, then
  - ➢ $x(A + B) = xA + xB$
  - ➢ $(x + y)A = xA + yA$
- If $A = [a_{ij}]$ is a matrix of order $m \times n$ and $B = [b_{jk}]$ is a matrix of order $n \times p$ then the product of the matrices A and B is a matrix C of order $m \times p$. It can be denoted as $AB = C = [C_{ik}]_{m\times p}$, where

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

- **Properties of multiplication of matrices** are as follows:
  - ➢ The associative law: If there are 3 matrices X, Y and Z we have $(XY)Z = X(YZ)$
  - ➢ Distributive law: If there are 3 matrices X, Y and Z then:

    $X(Y + Z) = XY + XZ$

    $(X + Y)Z = XZ + YZ$
  - ➢ The existence of multiplicative identity: For every square matrix X, there exists an identity matrix of the same order such that $IX = XI = X$.
  - ➢ Multiplication is not commutative: $AB \neq BA$

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

1 Mark Questions

1. Find the value of a if

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}.$$

[DELHI 2013]

2. If $\begin{bmatrix} x+1 & x-1 \\ x-3 & x+2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}$, then write the value of x.

[DELHI 2013]

3. If $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$, then find the matrix A.

[DELHI 2013]

4. Write the element a_{23} of a 3×3 matrix $A = (a_{ij})$ whose elements a_{ij} are given by $a_{ij} = \frac{|i-j|}{2}$.

[DELHI 2015]

5. If A is a 3×3 invertible matrix, then what will be the value of k if

$$\det(A^{-1}) = (\det A)^k .$$

[DELHI 2017]

6. If A is a square matrix such that $A^2 = I$, then find the simplified value of

$$(A-I)^3 + (A+I)^3 - 7A$$

[DELHI 2016]

7. If $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$, write the value of x.

[ALL INDIA 2012]

8. For a 2 × 2 matrix, A = $[a_{ij}]$, whose elements are given by $a_{ij} = \frac{i}{j}$, write the value of a_{12}

[DELHI 2011]

9. For what value of x the matrix $\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$ is singular?

[DELHI 2011]

10. Write the value of $x - y + z$ from the following equation:

$$\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$$

[ALL INDIA 2011]

11. Write the order of the product matrix:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$$

[ALL INDIA 2011]

12. Simplify:

$$\cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

[ALL INDIA 2012]

13. If matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $A^2 = kA$, then write the value of k.

[ALL INDIA 2013]

14. If $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$, find the value of x + y.

[ALL INDIA 2014]

15. If A is a square matrix such that $A^2 = A$, then write the value of $7A - (I + A)^3$, where I is an identity matrix. [4 marks]

[ALL INDIA 2014]

4 Marks Questions

16. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, find $A^2 - 5A + 4I$ and hence find a matrix X such that $A^2 - 5A + 4I + X = 0$.

[DELHI 2015]

17. Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$, find a matrix D such that $CD - AB = O$.

[DELHI 2017]

Solutions

1. $a - b = -1 \Rightarrow a + 1 = b$

$2a - b = 0$

$\Rightarrow 2a - (a+1) = 0 \qquad (\because a \neq b)$

$\Rightarrow 2a - a - 1 = 0$

$a = 1$ [1]

2. $(x+1)(x+2) - (x-1)(x-3) = 12 + 1$

$\therefore (x^2 + 2x + x + 2) - (x^2 - 3x - x + 3) = 13$

$\therefore 3x + 2 + 4x - 3 = 13$

$\therefore 7x - 1 = 13$

$\therefore 7x = 14$

$\therefore x = 2$ [1]

3. $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$

$\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix} = A$ [½]

$\begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix} = A$ [½]

4. $a_{ij} = \dfrac{|i-j|}{2}$

$a_{23} = \dfrac{|2-3|}{2} = \dfrac{1}{2}$

The element a_{23} is $\dfrac{1}{2}$. [1]

5. Given $\det(A^{-1}) = (\det A)^k$

$\det(A^{-1}) = \dfrac{1}{\det(A)}$

$= \dfrac{1}{|A|} = |A|^{-1} = (\det A)^{-1}$ [½]

From above two equation we get,

$(\det A)^{-1} = (\det A)^k$

Thus, $k = -1$ [½]

6. Using property $\left(a-b\right)^3 = a^3+b^3-3a^2b+3ab^2$

we get: $\left(\text{A-I}\right)^3+\left(\text{A+I}\right)^3\text{-7A}$

$= A^3 - I^3 - 3A^2I + 3AI^2 + A^3 + I^3 + 3A^2I + 3AI^2 - 7A$ [½]

$= 2A^3 + 6AI^2 - 7A$

$= 2A^2.A + 6AI.\text{I} - 7A$

$= 2I.A + 6AI.\text{I} - 7A$

$= 2A + 6A - 7A = A$ [½]

7. Given: $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$

$\therefore \begin{pmatrix} 2-6 & -6+12 \\ 5-14 & -15+28 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$

$\therefore \begin{pmatrix} -4 & 6 \\ -9 & 13 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$

$x = 13$ [1]

8. Given $a_{ij} = \dfrac{i}{j}$

When $i = 1,\ j = 2,\ a_{12} = \dfrac{1}{2}$ [1]

9. Matrix A is singular if $|A| = 0$

$\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$ is singular

$\therefore \begin{vmatrix} 5-x & x+1 \\ 2 & 4 \end{vmatrix} = 0$

$\therefore 4\left(5-x\right) - 2\left(x+1\right) = 0$

$\therefore 20 - 4x - 2x - 2 = 0$

$\therefore 18 = 6x$

$\therefore x = \dfrac{18}{6} = 3$ [1]

10. $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$

$x + y + z = 9$ ...(1)

$x + z = 5$ ...(2)

$y + z = 7$ ...(3)

From equation (1) and (2)

$y = 4$

From equation (1) and (3)

$x = 2$

Thus,

$x = 2, y = 4, z = 3$

$\therefore x - y = 7$ [1]

$= 2 - 4 + 3 = 1$

11. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$

$(3 \times 1) \times (1 \times 3) = (3 \times 3)$ matrix

12. $\begin{bmatrix} \cos^2\theta & \sin\theta.\cos\theta \\ -\sin\theta.\cos\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta & \sin^2\theta \end{bmatrix}$

$= \begin{bmatrix} \sin^2\theta + \cos^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ [1]

13. Given $A^2 = kA$

$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$

$\Rightarrow k = 2$ [1]

14. Given, $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$

Comparing the element of two matrix, we get

so $x - y = -1 \rightarrow (\text{i})$

$2x - y = 0 \rightarrow (\text{ii})$

On solving both eqn we get $x = 1,\ y = 2$

so $x + y = 3$ [1]

15. $A^2 = A$

$7A - (I + A)^3$

$$7A - \left[(I+A)^2(I+A)\right] = 7A - \begin{bmatrix}(1I + AA + 2AI)\\(I+A)\end{bmatrix}$$

$$= 7A - \left[I + A^2 + 2AI\right]\left[I + A\right]$$

$$= 7A - [I + A + 2A][I + A]$$

$$= 7A - [I + 3A][I + A]$$

$$= 7A - [I^2 + IA + 3AI + 3A^2]$$

$$= 7A - [I + A + 3A + 3A]$$

$$= 7A - [I + 7A]$$

= I. [1]

16. $$A^2 - 5A + 4I = \begin{bmatrix}2 & 0 & 1\\2 & 1 & 3\\1 & -1 & 0\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\2 & 1 & 3\\1 & -1 & 0\end{bmatrix} - 5\begin{bmatrix}2 & 0 & 1\\2 & 1 & 3\\1 & -1 & 0\end{bmatrix} + 4\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}4+0+1 & -1 & 2\\4+2+3 & 1-3 & 2+3\\2-2 & -1 & 1-3\end{bmatrix} - \begin{bmatrix}10 & 0 & 5\\10 & 5 & 15\\5 & -5 & 0\end{bmatrix} + \begin{bmatrix}4 & 0 & 0\\0 & 4 & 0\\0 & 0 & 4\end{bmatrix}$$

[1]

$$= \begin{bmatrix}5 & -1 & 2\\9 & -2 & 5\\0 & -1 & -2\end{bmatrix} - \begin{bmatrix}10 & 0 & 5\\10 & 5 & 15\\5 & -5 & 0\end{bmatrix} + \begin{bmatrix}4 & 0 & 0\\0 & 4 & 0\\0 & 0 & 4\end{bmatrix}$$ [1]

$$= \begin{bmatrix}5-10+4 & -1-0+0 & 2-5+0\\9-10+0 & -2-5+4 & 5-15+0\\0-5+0 & -1+5+0 & -2-0+4\end{bmatrix}$$ [1]

$$A^2 - 5A + 4I = \begin{bmatrix}-1 & -1 & -3\\-1 & -3 & -10\\-5 & 4 & 2\end{bmatrix} \quad ...(i)$$

Now,

$$A^2 - 5A + 4I + X = 0$$

From eqn. (i)

$$\therefore \begin{bmatrix}-1 & -1 & -3\\-1 & -3 & -10\\-5 & 4 & 2\end{bmatrix} + X = 0$$

$$\therefore X = -\begin{bmatrix}-1 & -1 & -3\\-1 & -3 & -10\\-5 & 4 & 2\end{bmatrix} \text{ or } \begin{bmatrix}1 & 1 & 3\\1 & 3 & 10\\5 & -4 & -2\end{bmatrix}$$ [1]

17. Given $A = \begin{pmatrix}2 & -1\\3 & 4\end{pmatrix}$, $B = \begin{pmatrix}5 & 2\\7 & 4\end{pmatrix}$, $C = \begin{pmatrix}2 & 5\\3 & 8\end{pmatrix}$, and

$CD - AB = O$

$\Rightarrow CD = AB$ ...(i)

$|C| = 16 - 15 = 1 \neq 0$ [1]

Thus inverse of C exist.

Taking C^{-1} on both sides of equation 1 we get

$C^{-1}CD = C^{-1}AB$

$ID = C^{-1}AB$

$D = C^{-1}AB$ ...(ii) [1]

Now, $AdjC = \begin{pmatrix}8 & -5\\-3 & 2\end{pmatrix}$

Formula $C^{-1} = \dfrac{1}{|C|}AdjC$

$C^{-1} = \dfrac{1}{1}\begin{pmatrix}8 & -5\\-3 & 2\end{pmatrix} = \begin{pmatrix}8 & -5\\-3 & 2\end{pmatrix}$ ....(iii) [1]

Using eq. (1) , (iii) and values of A, B we get

$$D = \begin{pmatrix}8 & -5\\-3 & 2\end{pmatrix}\begin{pmatrix}2 & -1\\3 & 4\end{pmatrix}\begin{pmatrix}5 & 2\\7 & 4\end{pmatrix}$$

$$= \begin{pmatrix}16-15 & -8-20\\-6+6 & 3+8\end{pmatrix}\begin{pmatrix}5 & 2\\7 & 4\end{pmatrix}$$

$$= \begin{pmatrix}1 & -28\\0 & 11\end{pmatrix}\begin{pmatrix}5 & 2\\7 & 4\end{pmatrix}$$

$$= \begin{pmatrix}5-196 & 2-112\\0+77 & 0+44\end{pmatrix}$$

$$D = \begin{pmatrix}-191 & -110\\77 & 44\end{pmatrix}$$ [1]

[TOPIC 2] Transpose of a Matrix and Symmetric and Skew Symmetric Matrices

Summary

- The **transpose of the matrix** $A = [a_{ij}]_{m\times n}$ is denoted by $A^T = [a_{ji}]_{n\times m}$ and is obtained by interchanging the rows with columns of matrix A.

  Example: If $A = \begin{bmatrix} -4 & 1 \\ 2 & 0 \end{bmatrix}$, then $A^T = \begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix}$.
- Some properties of transpose of the matrices are as follows:

  If A and B are matrices of suitable orders then
  - $(A^T)^T = A$
  - $(kA)^T = kA^T$ (Where k is any constant)
  - $(A+B)^T = A^T + B^T$
  - $(AB)^T = B^T A^T$
- If $A^T = A$ then the square matrix $A = [a_{ij}]$ is said to be **symmetric matrix** for all possible values of i and j.

  Example: $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 4 & 7 & 0 \end{bmatrix}$
- If $A^T = -A$ then the square matrix $A = [a_{ij}]$ is said to be **skew symmetric matrix** for all the possible values of i and j. All the diagonal elements of a skew symmetric matrix are zero.

  Example: $A = \begin{bmatrix} 1 & 2 & -4 \\ -2 & 3 & 7 \\ 4 & -7 & 0 \end{bmatrix}$ then

  $$A^T = \begin{bmatrix} 1 & -2 & 4 \\ 2 & 3 & -7 \\ -4 & 7 & 0 \end{bmatrix} = -\begin{bmatrix} 1 & 2 & -4 \\ -2 & 3 & 7 \\ 4 & -7 & 0 \end{bmatrix} = -A$$
- For any square matrix A with real number entries, $A + A^T$ is a symmetric matrix and $A - A^T$ is a skew symmetric matrix.
- Any square matrix can be expressed as the sum of a symmetric matrix and a skew symmetric matrix

  i.e $A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right)$

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 2

1 Mark Questions

1. Matrix $A = \begin{vmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{vmatrix}$ is given to be symmetric, find values of a and b.

[DELHI 2016]

2. If the matrix $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ a skew symmetric matrix, find the value of 'a' and 'b'?

[DELHI 2018]

3. For what value of x, is the matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$ a symmetric matrix?

[ALL INDIA 2013]

4. If A is a skew-symmetric matrix of order 3, then prove that det A = 0.

[ALL INDIA 2017]

5. If A = $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$, find $0 < \alpha < \frac{\pi}{2}$ satisfying $A + A^T = \sqrt{2} I_2$ when A^T is transpose of A.

[ALL INDIA 2016]

2 Marks Question

6. Show that all the diagonal elements of a skew symmetric matrix are zero.

[DELHI 2017]

Solutions

1. A is a symmetric matrix. $\Rightarrow A^T = A$

$$\begin{bmatrix} 0 & 3 & 3a \\ 2b & 1 & 3 \\ -2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$$

Therefore, $3a = -2$

$\Rightarrow a = \frac{-2}{3}$ [½]

Also, $2b = 3$

$\Rightarrow b = \frac{3}{2}$ [½]

2. Given that 'A' is skew symmetric matrix

$\Rightarrow A^T = -A$

$$\begin{bmatrix} 0 & 2 & b \\ a & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2 & 2 & b \\ a & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a & 3 \\ -2 & 0 & 1 \\ -b & -1 & 0 \end{bmatrix}$$ [½]

On comparing the above matrices we get,

$a = -2$

$b = 3$ [½]

3. The value of determinants of skew symmetric matrix of odd order is always equal to zero.

$$\begin{vmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{vmatrix} = 0$$

$-1(0-3x) - 2(3-0) = 0$

$\Rightarrow 3x - 6 = 0$

$\Rightarrow x = 2$ [1]

4. Let A= $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$ be a skew symmetric matrix of order 3.

$$\therefore |A| = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$$

$|A| = -a(0+bc) + b(ac-0)$ [1]

$= -abc + abc = 0$ **Proved.**

5. $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$

$$A^T = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$\Rightarrow A + A^T = \begin{pmatrix} \cos\alpha + \cos\alpha & 0 \\ 0 & 2\cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{pmatrix}$$

$\because A + A^T = \sqrt{2}I$

$$\therefore \begin{pmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$ [½]

By comparing : $2\cos\alpha = \sqrt{2}$

$\therefore \cos\alpha = \frac{1}{\sqrt{2}}$

So, $\alpha = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$ [½]

6. Let $A = [a_{ij}]$ be a skew symmetric matrix.

Thus $A^i = -A$

$a_{ii} = -a_{ii} \forall i, j$

$a_{ii} + a_{ii} = 0$

$2a_{ii} = 0$ [1]

$a_{ii} = 0$ for all values of i.

Thus, $a_{11} = 0, a_{22} = 0, a_{33} = 0 \ldots.. a_{nn} = 0$ [1]

[TOPIC 3] Inverse of matrices by Elementary row transformation

- Elementary operation of a matrix are as follows:
  - **Interchanging of two rows or columns:** $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$ represents that the ith row or column is interchanged with jth row or column.
  - **Multiplying the row or column of matrix by non- zero scalar:** $R_i \rightarrow lR_j$ or $C_i \rightarrow lC_j$ where I is any non- zero number, represents the ith row or column is multiplied by I.
  - **Adding the elements of any row or column to another row or column:** $R_i \rightarrow R_i + lR_j$ or $C_i \rightarrow C_i + lC_j$, where I is any non-zero number, represents that the jth row or column is multiplied by I and added to respective element of ith row or column.
- If X is a square matrix of order n and if there exists another square matrix Y of the same order n, such that $XY = YX = I$, then Y is called the inverse matrix of X and it is denoted by X^{-1}.
- The inverse of a matrix can be found using row or column operations.
- If Y is the inverse of X, then X is also the inverse of Y. Also, $(XY)^{-1} = Y^{-1}X^{-1}$
- Inverse of a square matrix, if it exists, is unique.

PREVIOUS YEARS' EXAMINATION QUESTIONS TOPIC 3

1 Mark Question

1. If $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$, write the minor of the element a_{23}.

[DELHI 2012]

6 Marks Questions

2. Use elementary transformations, find the inverse of the matrix $A = \begin{pmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

And use it to solve the following system of linear equations:

$8x + 4y + 3z = 19 \quad 2x + y + z = 5 \quad x + 2y + 2z = 7$

[DELHI 2016]

3. Using elementary transformations, find the inverse of the matrix $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

[DELHI 2011]

4. Using elementary operations, find the inverse of the following matrix:

$\begin{pmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$

[DELHI 2012]

5. Using elementary row transformations, find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$.

[DELHI 2018]

6. Find A-1 using row elementary operations, given that $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$.

[DELHI 2011]

Solutions

1. Given :Minor of the element

$$a_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= 10 - 3 = 7$$ [1]

2. $A = IA$

$$\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 8 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$ [½]

Applying $R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ 8 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 0 \end{bmatrix} A$$ [½]

Applying $R_3 \to R_3 - 8R_1$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & 12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying $R_2 \leftrightarrow \frac{-1}{3} R_2$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{-1}{3} & \frac{2}{3} \\ 1 & 0 & -8 \end{bmatrix} A$$ [½]

Applying $R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & \frac{-1}{3} \\ 0 & \frac{-1}{3} & \frac{2}{3} \\ 1 & 0 & -8 \end{bmatrix} A$$ [½]

Applying $R_3 \to R_3 + 12R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & \frac{-1}{3} \\ 0 & \frac{-1}{3} & \frac{2}{3} \\ 1 & -4 & 0 \end{bmatrix} A$$ [½]

Applying $R_3 \leftrightarrow -1R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & \frac{-1}{3} \\ 0 & \frac{-1}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix} A$$ [½]

Applying $R_2 \to R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & \frac{-1}{3} \\ 1 & \frac{-13}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix} A$$ [½]

$$I = A^{-1}A$$

$$A^{-1} = \begin{bmatrix} 0 & \frac{2}{3} & \frac{-1}{3} \\ 1 & \frac{-13}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix} \text{.....}(i)$$ [½]

Writing the equations in matrix form

$$\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 19 \\ 5 \\ 7 \end{bmatrix}$$

$AX = B$

$A^{-1}AX = A^{-1}B$ (Pre-Multiplying by A^{-1})

$IX = A^{-1}B$

$X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & \frac{-13}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 19 \\ 5 \\ 7 \end{bmatrix}$$ [½]

...From(i)

$$= \begin{bmatrix} 0 + \frac{10}{3} - \frac{7}{3} \\ 19 - \frac{65}{3} + \frac{14}{3} \\ -19 + 20 + 0 \end{bmatrix}$$ [½]

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$x = 1,\ y = 2,\ z = 1$ [1]

3. Let A = $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

$A = IA$

$$\therefore \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$ [1]

Applying $R_2 \to R_2 + 3R_1$

$$= \begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -7 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$ [½]

Applying $R_3 \to R_3 - 2R_1$

$$= \begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -7 \\ 0 & -5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$ [½]

Applying $R_2 \to \frac{1}{9} R_2$

$$= \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & \frac{-7}{9} \\ 0 & -5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$ [½]

Applying $R_1 \to R_1 - 3R_2$

$$= \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{7}{9} \\ 0 & -5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$ [½]

Applying $R_3 \to R_3 + 5R_2$

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{7}{9} \\ 0 & 0 & \frac{1}{9} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -\frac{1}{3} & \frac{5}{9} & 1 \end{bmatrix} A$$ [½]

Applying $R_3 \to 9R_3$

$$= \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{7}{9} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -3 & 5 & 9 \end{bmatrix} A$$ [½]

Applying $R_1 \to R_1 - \frac{1}{3}R_3$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{9} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -3 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -3 & 5 & 9 \end{bmatrix} A$$ [½]

Applying $R_2 \to R_2 + \frac{7}{9}R_3$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix} A$$ [½]

$I = A^{-1}A$

$$A^{-1} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix}$$ [1]

4. $A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Now, $A = IA$

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$ [1]

Using elementary row transformation $R_1 \to R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$ [½]

Applying $R_2 \to R_2 + R_1, R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$ [1]

Applying $R_2 \to \frac{1}{3}R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{5}{3} \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$ [½]

Applying $R_1 \to R_1 - 2R_2, R_3 \Rightarrow R_{3+}5R_2 5R_3$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{5}{3} & -\frac{4}{3} & 1 \end{bmatrix} A$$ [½]

Applying $R_3 \to 3R_3$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ 5 & -4 & 3 \end{bmatrix} A$$ [½]

Applying $R_1 \to R_1 + \frac{1}{3}R_3, R_2 \to R_2 - \frac{5}{3}R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix} A$$ [1]

$I = A^{-1}A$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix}$$ [1]

5. Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$

As we have to apply row operations so
A = IA

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$ [1]

Applying $R_3 \to R_3 - R_1, \quad R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$ [2]

Applying $R_2 \to R_2 - R_3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$ [1]

Applying $R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$ [1]

$I = A^{-1}A$

$$A^{-1} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$ [1]

6. $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

$A = IA$

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$ [1]

$R_1 \to \frac{1}{2} R_1$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$ [½]

$R_2 \to R_2 - 5R_1$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{5}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$ [½]

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{5}{2} & 1 & 0 \\ \frac{5}{2} & -1 & 1 \end{bmatrix} A$$ [½]

$R_3 \to 2R_3$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{5}{2} & 1 & 0 \\ 5 & -2 & 2 \end{bmatrix} A$$ [½]

$R_1 \to R_1 + \frac{1}{2} R_3$

$R_2 \to R_2 - \frac{5}{2} R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A$$ [2]

$$A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$ [1]

Value Based Questions

PREVIOUS YEARS'

EXAMINATION QUESTIONS

4 Marks Questions

1. To raise money for an orphanage, students of three schools A, B and C organized an exhibition in their locality, where they sold paper bags, scrap-books and pastel sheets made by them using recycled paper, at the rate of Rs. 20, Rs. 15 and Rs. 5 per unit respectively. School A sold 25 paper bags, 12 scrap-books and 34 pastel sheets. School B sold 22 paper bags, 15 scrap-books and 28 pastel sheets while School C sold 26 paper bags, 18 scrap-books and 36 pastel sheets. Using matrices, find the total amount raised by each school. By such exhibition, which values are generated in the students?

[ALL INDIA 2015]

2. Three schools A, B and C organized a mela for collecting funds for helping the rehabilitation of flood victims. They sold handmade fans, mats and plates from recycled material at a cost of 25, 100 and 50 each. The numbers of articles sold are given below:

School / Article	A	B	C
Hand-Fans	40	25	35
Mats	50	40	50
Plates	20	30	40

Find the funds collected by each school separately by selling the above articles. Also find the total funds collected for the purpose.

[DELHI 2015]

Solutions

1. $\begin{bmatrix}20 & 15 & 5\end{bmatrix}\begin{pmatrix}25 & 12 & 34\\ 22 & 15 & 28\\ 26 & 18 & 36\end{pmatrix} = \begin{pmatrix}650\\ 805\\ 970\end{pmatrix}$ [1]

$= [20 \times 15 + 15 \times 12 + 5 \times 3420 \times 22 + 15 \times 15 + 5 \times 2820 \times 26 + 15 \times 18 + 5 \times 36$

$= [650 \quad 805 \quad 970]$ [1]

Thus the total amount generated

= Rs.650 + Rs. 805 + Rs. 970 = Rs. 2425 [1]

The value generated in above question is 'Humanity'. [1]

2. They sold handmade fans, mats and plates, at a cost Rs. 25, Rs. 100 and Rs. 50 each.

$A = \begin{bmatrix}25\\ 100\\ 50\end{bmatrix}$

The number of articles sold

$B = \begin{bmatrix}40 & 50 & 20\\ 25 & 40 & 30\\ 35 & 50 & 40\end{bmatrix}$

Funds collected = BA $= \begin{bmatrix}40 & 50 & 20\\ 25 & 40 & 30\\ 35 & 50 & 40\end{bmatrix}\begin{bmatrix}25\\ 100\\ 50\end{bmatrix}$ [1]

$= \begin{bmatrix}100+5000+1000\\ 625+4000+1500\\ 875+5000+2000\end{bmatrix} = \begin{bmatrix}7000\\ 6125\\ 7875\end{bmatrix}$ [1]

The funds collected by school $A = 7000$

The funds collected by school $B = 6125$

The funds collected by school $C = 7875$

Total funds collected for the purpose = 21000 [1]

Values:

(i) Helpful and caring nature of students.

(ii) Use of recycled material, i.e. helping in conserving the environment. [1]

Smart Notes ....

Smart Notes ....

CHAPTER 4

Determinants

Chapter Analysis of Previous Three Years' Board Exams

Number of Questions Asked in Exams

Topics	2018		2017		2016	
	Delhi	**All India**	**Delhi**	**All India**	**Delhi**	**All India**
Value of Determinant			4 marks		1 marks	1 marks
Property of Determinant	4 marks			4 marks		6 marks

[TOPIC 1] Expansion of Determinants

Summary

- **Definition**: A determinant is a number (real or complex) that can be related to any square matrix A = [a_{ij}] of order *n*. It is denoted as det(A).

  Determinant of a matrix $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ can be given as:

  $$|A|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$$

  Determinant of a matrix $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ by expanding along R_1 can be given as:

  $$\det(A)=|A|=(-1)^{1+1}a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

  $$+(-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+(-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- **Minors:** The minor M_{ij} of a_{ij} in A is the determinant of the square sub matrix of order $(n-1)$ obtained by deleting its i^{th} row and j^{th} column in which a_{ij} lies. It is denoted by M_{ij}.

  Minor of an element of a determinant of order n(for all $n \geq 2$) is a determinant of order $n-1$.

- **Co-factors:** Co-factor of an element a_{ij} is defined by $A_{ij}=(-1)^{(i+j)}M_{ij}$, where M_{ij} is a minor of a_{ij}. Co-factor is denoted by A_{ij}.

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

1 Mark Questions

1. If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix}=\begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, Find the value of x.

[ALL INDIA 2014]

2. If A_{ij} is the cofactor of element a_{ij} of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$, then write the value of $a_{32}.A_{32}$.

[ALL INDIA 2013]

3. $\begin{vmatrix} x & x \\ 1 & x \end{vmatrix}=\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$, write the positive value of x

[ALL INDIA 2011]

4. If $\Delta=\begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$, write the minor of the elements a_{23}.

[ALL INDIA 2012]

5. Simplify:

$$\cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}+\sin\theta\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

[DELHI 2012]

6. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix}=\begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, write the value of x.

[DELHI 2014]

7. Find the maximum value of

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$$

[DELHI 2016]

Solutions

1. $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$

On expanding both determinants we get

12x + 14 = 32 – 42

12x + 14 = –10

12x = –24

x = – 2 [1]

2. $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

$A_{32} = (-1)^{3+2}\, M_{32}$ *is the minor of* a_{32}

$A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix}$

$A_{32} = -\begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix}$

$\Rightarrow A_{32} = -(8-30)$

$A_{32} = 22$

$\therefore\ a_{32} A_{32} = 5(22) = 110$ [1]

3. $x^2 - x = 6 - 4$

$x^2 - x - 2 = 0$

$x^2 - 2x + x - 2 = 0$

x(x – 2) + 1(x – 2)

(x + 1)(x – 2) = 0

x = 2, –1

Positive value of x = 2 [1]

4. Minor of $a_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7$ [1]

5. $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ [1]

6. $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$

$2x^2 - 40 = 18 + 14$

$2x^2 = 32 + 40 = 72$

$\Rightarrow x^2 = \frac{72}{2} = 36$

$\therefore\ x = \pm 6$ [1]

7. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$

By $R_1 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1+\sin\theta & 0 \\ 0 & 0 & \cos\theta \end{vmatrix}$ [½]

Expanding along C_1, we get:

$= 1(\sin\theta\cos\theta)$

$= \frac{1}{2}(2\sin\theta\cos\theta) = \frac{1}{2}\sin 2\theta$

Therefore, maximum value = $\frac{1}{2} \times 1 = \frac{1}{2}$ [½]

[TOPIC 2] Properties of Determinants

Summary

- **Properties of determinants**:
  - The value of the determinant remains unchanged if its rows and columns are interchanged.
  - If A is a square matrix, then $det(A) = det(A')$, where A'= transpose of A.
  - If any two columns (or rows) of a determinant are interchanged, then the sign of the determinant changes.
  - The determinant of the product of two square matrices of same order is equal to the product of their respective determinants, that is $|AB| = |A||B|$.
  - If any two rows or columns of a determinant are identical (i.e. all corresponding elements are same), then value of determinant is zero.
  - If each element of a row or a column of a determinant is multiplied by a constant k, then its value gets multiplied by k.
  - If some or all elements of a row or column of a determinant are expressed as sum of two or more terms, then the determinant can be expressed as sum of two or more determinants.

    For example:

$$\begin{vmatrix} a_1+x & a_2+y & a_3+z \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

  - If we multiply each element of a row or a column of a determinant, by a constant k, then the value of the determinant is also multiplied by k.
  - Multiplying a determinant by k means multiply elements of any one row or any one column by k.
  - Adding or subtracting each element of anycolumn or any row of a determinant withthe equimultiples of corresponding elements of any other row or column, does not changes the value of the determinant, i.e., the value of determinant remain same if we apply the operation $R_i \to R_i \to kR_j$.
- **Area of triangle:** If a triangle is given with its vertices at points (x_1, y_1), (x_2, y_2) and (x_3, y_3), then its area can be calculated as:

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

  - Area is a positive quantity, so we always take the absolute value of the determinant.
  - If the area of the triangle is already given, use both positive and negative values of the determinant for calculation.
  - Area of triangle formed by three collinear points is always zero.

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 2

1 Mark Question

1. If A is a 3 × 3 matrix and | 3A | = k | A |, then write the value of k.

[ALL INDIA 2016]

4 Marks Questions

2. Using the properties of determinants, prove the following :

$$\begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(1-x) & x(x-1)(x-2) & x(x+1)(x-1) \end{vmatrix}$$

$$= 6x^2(1-x^2)$$

[ALL INDIA 2015]

3. Using properties of determinants, prove the following:

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = \left(1-x^3\right)^2.$$

[DELHI 2013]

4. Prove using properties of determinants:

$$\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$$

[ALL INDIA 2011]

5. If $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, using properties of determinants find the value of $f(2x) - f(x)$.

[DELHI 2015]

6. Using properties of determinants, prove that

$$\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} = (x+y+z)^3$$

[DELHI 2014]

7. Using properties of determinants, prove that.

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

[ALL INDIA 2012]

8. Using properties of determinants, prove that

$$\begin{bmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{bmatrix} = 4a^2b^2c^2$$

[DELHI 2011]

9. Using properties of determinant, prove that

$$\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix} = 9(3xyz+xy+yz+zx)$$

[DELHI 2018]

10. Using properties of determinants, prove the following:

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y)$$

[ALL INDIA 2013, 2017]

11. Using properties of determinants, prove that

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$$

[ALL INDIA 2014]

12. Using properties of determinant prove that

$$\Delta = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$$

[ALL INDIA 2017]

6 Marks Questions

13. Prove that $\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$ is divisible by $(x+y+z)$ and hence find the quotient.

[DELHI 2016]

14. Using properties of determinant, prove that

$$\begin{vmatrix} (x+y)^2 & zx & zy \\ zx & (z+y)^2 & xy \\ zx & xy & (z+x)^2 \end{vmatrix} = 2xyz(x+y+z)^3$$

[ALL INDIA 2016]

Solutions

1. A is a matrix of 3 × 3 say

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$3A = \begin{vmatrix} 3a_{11} & 3a_{12} & 3a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{vmatrix}$$

$\therefore |3A| = 3 \times 3 \times 3|A| = 27\,|A|$

Which is given as $k|A|$ so k = 27 [1]

2. Taking x from R_2, $x(x-1)$ from R_3 and $(x+1)$ from C_3

$$\Delta = x^2(x-1)(x+1)\begin{vmatrix}1 & x & 1\\ 2 & x-1 & 1\\ -3 & x-2 & 1\end{vmatrix}$$ [2]

$$C_2 \to C_2 - xC_1,\ C_3 \to C_3 - C_1$$

$$x^2(x^2-1)\begin{vmatrix}1 & 0 & 0\\ 2 & -x-1 & -1\\ -3 & 4x-2 & 4\end{vmatrix}$$

$$= x^2(x^2-1)\begin{vmatrix}-1(1+x) & -1\\ 4x-2 & 4\end{vmatrix}$$

$$6x^2(1-x^2)$$ [2]

3. Given: $L.H.S. = \begin{vmatrix}1 & x & x^2\\ x^2 & 1 & x\\ x & x^2 & 1\end{vmatrix}$

$$= \begin{vmatrix}1+x+x^2 & x & x^2\\ 1+x+x^2 & 1 & x\\ 1+x+x^2 & x^2 & 1\end{vmatrix}$$ [½]

$$= (1+x+x^2)\begin{vmatrix}1 & x & x^2\\ 1 & 1 & x\\ 1 & x^2 & 1\end{vmatrix}$$ [½]

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$= (1+x+x^2)\begin{vmatrix}1 & x & x^2\\ 0 & 1-x & x-x^2\\ 0 & x^2-x & 1-x^2\end{vmatrix}$$ [1]

$$= (1+x+x^2)\begin{vmatrix}1 & x & x^2\\ 0 & 1-x & x(1-x)\\ 0 & x(x-1) & (1+x)(1-x)\end{vmatrix}$$

Taking common $(1-x)$ each from R_2 and R_3 respectively

$$= (1-x)^2(1+x+x^2)\begin{vmatrix}1 & x & x^2\\ 0 & 1 & x\\ 0 & -x & 1+x\end{vmatrix}$$ [½]

$$= (1-x)^2(1+x+x^2).\left[(1+x)+x^2\right]$$

Expanding along C_1

$$= \left[(1-x)(1+x+x^2)\right]^2$$ [½]

$$= (1-x^3)^2$$

$$= R.H.S\left[(a-b)(a^2+ab+b^2) = a^3-b^3\right]$$

4. LHS

$$= \begin{vmatrix}y+k & y & y\\ y & y+k & y\\ y & y & y+k\end{vmatrix}$$

$$R_1 \to R_1 + R_2 + R_3$$

$$= \begin{vmatrix}3y+k & 3y+k & 3y+k\\ y & y+k & y\\ y & y & y+k\end{vmatrix}$$ [1½]

$$= (3y+k)\begin{vmatrix}1 & 1 & 1\\ y & y+k & y\\ y & y & y+k\end{vmatrix}$$

$$C_2 \to C_2 - C_1$$

$$C_3 \to C_3 - C_1$$

$$= (3y+k)\begin{vmatrix}1 & 0 & 0\\ y & k & 0\\ y & 0 & k\end{vmatrix}$$ [2]

$$= (3y+k)(k^2-0)$$

$$= k^2(3y+k)$$ [½]

= RHS

5. We have, $f(x)=\begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, [½]

Applying $R_2 \to R_2 - xR_1$ and $R_3 \to R_3 - x^2 R_1$

$$f(x)=\begin{vmatrix} a & -1 & 0 \\ 0 & a+x & -1 \\ 0 & ax+x^2 & a \end{vmatrix},$$ [1]

Expanding along C_1, we have,

$$f(x) = a\left[a(a+x)+1\left(ax+x^2\right)\right]$$

$$= a\left(a^2 + ax + ax + x^2\right)$$ [1]

$$= a\left(a^2 + 2ax + x^2\right)$$

$$= a(x+a)^2 \quad \text{...(i)}$$

$$\therefore\ f(2x) = a(2x+a)^2 \quad \text{...(ii)}$$ [½]

Now, $f(2x) - f(x) = a(2x+a)^2 - a(x+a)^2$

$$= a\left[(2x+a)^2 - (x+a)^2\right]$$

$$= a(2x+a+x+a)(2x+a-x-a)$$ [1]

$$= ax(3x+2a)$$

6. LHS = $\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$

$$=\begin{vmatrix} x+y+z & x+y+z & x+y+z \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$ [1]

Taking common (x+y+z) from R_1, we have

$$=(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$ [1]

Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$ we get,

$$=(x+y+z)\begin{vmatrix} 1 & 0 & 0 \\ 2z & 0 & -(x+y+z) \\ x-y-z & x+y+z & x+y+z \end{vmatrix}$$ [1]

Expanding along R_1

$$= (x+y+z)(x+y+z)^2$$

$$= (x+y+z)^3 = \text{RHS}$$ [1]

7. Given: $L.H.S = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$

Applying $R_1 \to R_1 - R_2 - R_3$

$$=\begin{vmatrix} -2a & -2p & -2x \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$ [1]

$$=-2\begin{vmatrix} a & p & x \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$ [1]

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$=-2\begin{vmatrix} a & p & x \\ c & r & z \\ b & q & y \end{vmatrix}$$ [1]

Interchanging R_2 and R_3

$$=2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$ [1]

= R.H.S

8. Given $L.H.S. = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$

Take a,b,c, common from R_1, R_2 and R_3 respectively

$$= abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$ [1]

Take a,b,c, common from C_1, C_2 and C_3 respectively

$$a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$ [1]

Applying n $C_1 \to C_1 + C_2$,

$$= a^2b^2c^2\begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$ [1]

Expanding along C_1

$$= a^2 b^2 c^2 .2(1+1)$$

$$4a^2b^2c^2 = R.H.S$$ [1]

9. Given L.H.S $\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & 1 & 1+3x \\ 3y & 0 & -3x \\ 0 & 3z & -3x \end{vmatrix}$$ [1]

Taking out 3 common from R_2 and R_3

$$= 9\begin{vmatrix} 1 & 1 & 1+3x \\ y & 0 & -x \\ 0 & z & -x \end{vmatrix}$$ [1]

Expanding along C_1.

$$= 9\left[1\begin{vmatrix} 0 & -x \\ z & -x \end{vmatrix} - y\begin{vmatrix} 1 & 1+3x \\ z & -x \end{vmatrix} + 0\begin{vmatrix} 1 & 1+3x \\ 0 & -x \end{vmatrix}\right]$$ [1]

$$= 9\left[1(zx) - y(-x - z - 3xz)\right]$$

$$= 9(zx + xy + yz + 3xyz)$$

$$= 9(3xyz + xy + yz + zx)$$ [1]

=R.H.S

Proved.

10. $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y)$

<u>LHS</u>

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$

Now, Applying $C_1 \to C_1 + C_2 + C_3$

$$\begin{vmatrix} 3x+3y & x+y & x+2y \\ 3x+3y & x & x+y \\ 3x+3y & x+2y & x \end{vmatrix}$$ [1]

$$3(x+y)\begin{vmatrix} 1 & x+y & x+2y \\ 1 & x & x+y \\ 1 & x+2y & x \end{vmatrix}$$ [1]

Now, Applying $R_1 \to R_1 - R_3, R_2 \to R_2 - R_3$

$$3(x+y)\begin{vmatrix} 0 & -y & 2y \\ 0 & -2y & y \\ 1 & x+2y & x \end{vmatrix}$$ [1]

$$3(x+y)\begin{vmatrix} -y & 2y \\ -2y & y \end{vmatrix}$$

$$3y^2(x+y)\begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix}$$

$$3y^2(x+y)(-1+4) = 9y^2(x+y).$$ [1]

Hence proved.

11. LHS $= \begin{vmatrix} x & x & x \\ 5x & 4x & 2x \\ 10x & 8x & 3x \end{vmatrix} + \begin{vmatrix} y & x & x \\ 4y & 4x & 2x \\ 8y & 8x & 3x \end{vmatrix}$

Taking x common from row one of first matrices, y common from first column of second matrices and x common from both second and third column in second matrices.

$$= x^3\begin{vmatrix} 1 & 1 & 1 \\ 5 & 4 & 2 \\ 10 & 8 & 3 \end{vmatrix} + yx^2\begin{vmatrix} 1 & 1 & 1 \\ 4 & 4 & 2 \\ 8 & 8 & 3 \end{vmatrix}$$ [1]

Applying $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$ in the first determinant, As, the first two columns of the 2^{nd} determinant are same

$$= x^3\begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 2 \\ 2 & 5 & 3 \end{vmatrix} + yx^2 \times 0$$ [2]

Expanding the first determinant through R_1

$$= x^3\left[1\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}\right]$$

$$= x^3(5-4)$$ [1]

= x^3 = RHS

12. Let $\Delta = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$

Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$

$$\Delta = \begin{vmatrix} a^2+2a-3 & 2a-2 & 0 \\ 2a-2 & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$ [1]

$$\Delta = \begin{vmatrix} (a+3)(a-1) & 2(a-1) & 0 \\ 2(a-1) & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$ [1]

$$\Delta = (a-1)^2 \begin{vmatrix} a+3 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$ [1]

$$\Delta = (a-1)^2 \left[0-0+1X(a+3-4)\right]$$

$$\Delta = (a-1)^3$$ [1]

13. Let $\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} -x^2-y^2-z^2+xy+yz+zx & zx-y^2 & xy-z^2 \\ -x^2-y^2-z^2+xy+yz+zx & xy-z^2 & yz-x^2 \\ -x^2-y^2-z^2+xy+yz+zx & yz-x^2 & zx-y^2 \end{vmatrix}$$

[1]

Taking common $-x^2-y^2-z^2+xy+yz+zx$ from C_1, we get:

$$= -(x^2+y^2+z^2-xy-yz-zx)\begin{vmatrix} 1 & zx-y^2 & xy-z^2 \\ 1 & xy-z^2 & yz-x^2 \\ 1 & yz-x^2 & zx-y^2 \end{vmatrix}$$

[1]

Apply $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$ we get:

$$= -(x^2+y^2+z^2-xy-yz-zx)$$

$$\begin{vmatrix} 1 & zx-y^2 & xy-z^2 \\ 0 & xy-z^2-zx+y^2 & yz-x^2-xy+z^2 \\ 0 & yz-x^2-zx+y^2 & zx-y^2-xy+z^2 \end{vmatrix}$$ [1]

$\because yz-x^2-zx+y^2$

$= z(y-x)+(y^2-x^2)$

$= (y-x)(x+y+z)$ ...(i)

$\because xy-z^2-zx+y^2$

$= x(y-z)+(y^2-z^2)$

$= (y-z)(y+z+x)$ ...(ii)

$\because yz-x^2-xy+z^2$

$= y(z-x)+(z^2-x^2)$

$= (z-x)(y+z+x)$ ...(iii)

From (i) , (ii) and (iii), we get:

$$= -(x^2+y^2+z^2-xy-yz-zx)$$

$$\begin{vmatrix} 1 & zx-y^2 & xy-z^2 \\ 0 & (y-z)(x+y+z) & (z-x)(x+y+z) \\ 0 & (y-x)(x+y+z) & -(y-z)(x+y+z) \end{vmatrix}$$

Taking common $(x+y+z)$ each from R_2 and R_3 respectively:

$$= -(x+y+z)^2(x^2+y^2+z^2-xy-yz-zx)$$

$$\begin{vmatrix} 1 & zx-y^2 & xy-z^2 \\ 0 & y-z & z-x \\ 0 & y-x & -(y-z) \end{vmatrix}$$ [1]

Expand along C_1, we get:

$$-(x+y+z)^2(x^2+y^2+z^2$$
$$-xy-yz-zx)[-(y-z)^2-(z-x)(y-x)]$$

$$=(x+y+z)(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$
$$[(y-z)^2+(z-x)(y-x)]$$

$$=(x+y+z)(x^3+y^3+z^3-3xyz)$$
$$[y^2+z^2-2yz+zy-xz-xy+x^2]$$

$$=(x+y+z)(x^3+y^3+z^3-3xyz)$$

$$\times\frac{2}{2}[x^2+y^2+z^2-xy-yz-zx]$$

$$\frac{1}{2}(x+y+z)(x^3+y^3+z^3-3xyz)$$

$$\times[2x^2+2y^2+2z^2-2xy-2yz-2zx]$$

$$=\frac{1}{2}(x+y+z)(x^3+y^3+z^3-3xyz)$$

$$\times[x^2+x^2+y^2+y^2+z^2+z^2-2xy-2yz-2zx]$$

$$=\frac{1}{2}(x+y+z)(x^3+y^3+z^3-3xyz)[(x^2+y^2-2xy)$$

$$+(y^2+z^2-2yz)+(z^2+x^2-2zx)$$ [1]

$$\Rightarrow \Delta=\frac{1}{2}(x+y+z)(x^3+y^3+z^3-3xyz)$$

$$[(x-y)^2+(y-z)^2+(z-x)^2]$$

Hence, given determinant (Δ) is divisible by $(x+y+z)$ and the quotient is

$$\frac{1}{2}(x+y+z)(x^3+y^3+z^3-3xyz)$$

$$[(x-y)^2+(y-z)^2+(z-x)^2]$$ [1]

14. $$\begin{vmatrix}(x+y)^2 & zx & zy\\ zx & (z+y)^2 & xy\\ zy & xy & (z+x)^2\end{vmatrix}$$

$$=\frac{xyz}{xyz}\begin{vmatrix}(x+y)^2 & z^2 & z^2\\ x^2 & (z+y)^2 & x^2\\ y^2 & y^2 & (z+x)^2\end{vmatrix}$$ [1]

Taking z, x and y common from, C_1, C_2, C_3 respectively, we get

$$C_1\to C_1-C_3, C_2\to C_2-C_3$$

$$\begin{vmatrix}(x+y)^2-z^2 & 0 & z^2\\ 0 & (z+y)^2-x^2 & x^2\\ y^2 & y^2 & (z+x)^2\end{vmatrix}$$ [1]

$$=\begin{vmatrix}(x+y+z)(x+y-z) & 0 & z^2\\ 0 & (z+y-x)(z+y+x) & x^2\\ (y-z-x)(y+z+x) & (y-z-x)(y+z+x) & (z+x)^2\end{vmatrix}$$

$$=(x+y+z)^2\begin{vmatrix}x+y-z & 0 & z^2\\ 0 & z+y-x & x^2\\ y-z-x & y-z-x & (z+x)^2\end{vmatrix}$$

$$R_3\to R_3-R_2-R_1$$

$$=(x+y+z)^2\begin{vmatrix}x+y-z & 0 & z^2\\ 0 & z+y-x & x^2\\ -2x & -2z & 2zx\end{vmatrix}$$ [1]

$$C_1\to C_1+\frac{1}{z}C_3,\ C_2\to C_2+\frac{1}{x}C_3$$

$$=(x+y+z)^2\begin{vmatrix}x+y & \frac{z^2}{x} & z^2\\ \frac{x^2}{x} & z+y & x^2\\ 0 & 0 & 2zx\end{vmatrix}$$ [1]

Expanding along R_3

$$=2xz(x+y+z)^2\left((x+y)(z+y)-\frac{x^2}{z}\cdot\frac{z^2}{x}\right)$$ [1]

$$=2xz(x+y+z)^2\left(xz+xy+yz+y^2-xz\right)$$

$$=2xyz(x+y+z)^3$$ [1]

[TOPIC 3] Adjoint and Inverse of a Matrix

Summary

- **Adjoint of a square matrix:**

  The adjoint of a square matrix $A = [a_{ij}]_{n\times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n\times n}$, where A_{ij} is the cofactor of the element a_{ij}. Adjoint of the matrix A is denoted by "$adj(A)$".

  Example: $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ then, $adj\,A = \begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix}$

  If A be any given square matrix of order n, then $A(adj\,A) = (adj\,A)A = AI$, where I is the identity matrix of order n.

- **Inverse of a Matrix**
  - **Singular matrix:** It is a matrix with zero determinant value. i.e. $|A| = 0$.
  - **Non-singular matrix:** It is a matrix with a non-zero determinant value. i.e. $|A| \neq 0$.

    If A and B are nonsingular matrices of the same order, then AB and BA are also nonsingular matrices of the same order.
  - A square matrix A is invertible if and only if A is nonsingular matrix.
  - If A is a nonsingular matrix, then its **inverse** exists which is given by $A^{-1} = \frac{1}{|A|} adj(A)$
- **Consistent system:** A system of equations is said to be consistent if there exist one or more solution to the system of equation.
- **Inconsistent system:** If the solution to the system of equation does not exist, then it is termed as inconsistent system.
- For the square matrix A in the matrix equation $AX = B$
  - $|A| \neq 0$, there exists unique solution. The system of equation is consistent.
  - $|A| = 0$ and $(adj\,A)B \neq 0$ then there exists no solution. The system is inconsistent.
  - $|A| = 0$ and $(adj\,A)B = 0$, then system may or may not be consistent.

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 3

1 Mark Questions

1. If for any 2 × 2 square matrix A, A(adj A) $= \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$, then write the value of $|A|$.

   [ALL INDIA 2017]

2. Write A^{-1} for $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$.

   [DELHI 2011]

3. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, then write A^{-1}.

   [ALL INDIA 2015]

4. For what values of k, the system of linear equations

   $x + y + z = 2$

   $2x + y - z = 3$

   $3x + 2y + kz = 4$ has a unique solution ?

   [ALL INDIA 2016]

2 Marks Question

5. Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1} and show that $2A^{-1} = 9I - A$.

   [DELHI 2018]

4 Marks Questions

6. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$, find $(A')^{-1}$.

   [DELHI 2015]

7. If $A = \begin{vmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{vmatrix}$, find $A^2 - 5A + 16I$.

   [ALL INDIA 2015]

8. If $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$ and $A^3 - 6A^2 + 7A + kI_3 = 0$.

   Find k.

   [ALL INDIA 2016]

6 Marks Questions

9. Using matrices,solve the following system of linear equation :

$x-y+2z=7,$

$3x+4y-5z=-5,$

$2x-y+3z=12$

[DELHI 2012]

10. Find O matrix 'A' such that

$$\begin{bmatrix}2 & -1\\1 & 0\\-3 & 4\end{bmatrix}A=\begin{bmatrix}-1 & -8\\1 & -2\\9 & 22\end{bmatrix}$$

[ALL INDIA 2017]

11. Solve system of linear equations, using matrix method:

$x-y+z=4$

$2x+y-3z=0$

$x+y+z=2$

[DELHI 2011]

12. Use product $\begin{bmatrix}1 & -1 & 2\\0 & 2 & -3\\3 & -2 & 4\end{bmatrix}\begin{bmatrix}-2 & 0 & 1\\9 & 2 & -3\\6 & 1 & -2\end{bmatrix}$ to solve the system of equations

$x+3z=9, -x+2y-2z=4, \quad 2x-3y+4z=-3.$

[DELHI 2017]

13. Using matrix method ,solve the following system of equations:

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4, \frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1, \frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2; x,y,z\neq 0$$

[DELHI 2011]

14. If $A=\begin{bmatrix}2 & -3 & 5\\3 & 2 & -4\\1 & 1 & -2\end{bmatrix}$ find A^{-1}. Use it to solve the system of equations. $\begin{matrix}2x-3y+5z=11\\3x+2y-4z=-5\\x+y-2z=-3\end{matrix}$.

[DELHI 2018]

15. Determine the product $\begin{bmatrix}-4 & 4 & 4\\-7 & 1 & 3\\5 & -3 & -1\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\1 & -2 & -2\\2 & 1 & 3\end{bmatrix}$ and use it to solve the system of equations $x-y+z=4, x-2y-2z=9, 2x+x+3x=1$

[CBSE 2017]

Solutions

1. By using property

$A(\text{adj } A)=|A|I_n$ [½]

$\Rightarrow |A|\, I_0=\begin{bmatrix}8 & 0\\0 & 8\end{bmatrix}$

$\Rightarrow |A|\, I_0=8\begin{bmatrix}8 & 0\\0 & 8\end{bmatrix}$

$\Rightarrow |A|\, I_0=8\begin{bmatrix}8 & 0\\0 & 8\end{bmatrix}$

$\Rightarrow 8$ [½]

2. Given $|A| = 6-5=1$ [½]

$adj\, A=\begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$

$A^{-1}=\frac{1}{|A|}adjA$

$=\frac{1}{1}\begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}=\begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$ [½]

3. $adjA=\begin{bmatrix}-2 & 5\\3 & 2\end{bmatrix}$

$A^{-1}=\frac{1}{|A|}adj(A)$

$|A|=-19$

$A^{-1}=-\frac{1}{19}\begin{pmatrix}-2 & -5\\-3 & 2\end{pmatrix}$ [1]

4. Let $x+y+z=2$ (1)

$2x+y-z=3$ (2)

$3x+2y+kz=4$ (3)

The system of linear equation has unique solution then

$$\begin{vmatrix}1 & 1 & 1\\2 & 1 & -1\\3 & 2 & k\end{vmatrix}\neq 0$$

$\Rightarrow 1(k+2)-1(2k+3)+1(4-3)\neq 0$

$\Rightarrow k+2-2k-3+1\neq 0$

$\Rightarrow k\neq 0$ [1]

So values of $k=\mathrm{R}-\{0\}$

5. Given $A=\begin{bmatrix}2 & -3\\ -4 & 7\end{bmatrix}$

To show $2A^{-1} = 9I - A$

Proceeding R.H.S

$= 9I - A$

$= 9\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}2 & -3\\ -4 & 7\end{bmatrix}$

$= \begin{bmatrix}9 & 0\\ 0 & 9\end{bmatrix}-\begin{bmatrix}2 & -3\\ -4 & 7\end{bmatrix}$ [½]

$= \begin{bmatrix}7 & 3\\ 4 & 2\end{bmatrix}$ ...(1)

Now finding A^{-1}

$|A| = 14 - 12 = 2 \neq 0$

$\Rightarrow A^{-1}$ exists.

$A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ then $adj\,A=\begin{bmatrix}a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{bmatrix}$

$adj\,A=\begin{bmatrix}7 & 3\\ 4 & 2\end{bmatrix}$ [½]

$A^{-1} = \frac{1}{|A|}\bullet adj\,A$

$A^{-1} = \frac{1}{2}\begin{bmatrix}7 & 3\\ 4 & 2\end{bmatrix}$

Proceeding L.H.S

$= 2A^{-1}$

$= 2\times\frac{1}{2}\begin{bmatrix}7 & 3\\ 4 & 2\end{bmatrix}$

$= \begin{bmatrix}7 & 3\\ 4 & 2\end{bmatrix}$ ...(2)

From (1) and (2),

L.H.S = R.H.S [1]

Proved.

6. $A=\begin{bmatrix}1 & -2 & 3\\ 0 & -1 & 4\\ -2 & 2 & 1\end{bmatrix}$,

$(A')=\begin{bmatrix}1 & 0 & -2\\ -2 & -1 & 2\\ 3 & 4 & 1\end{bmatrix}$ [1]

$|A'| = 1(-1-8)-2(-8+3) = -9+10 = 1 \neq 0$ [½]

(A′) is a non singular matrix.

$(A')^{-1}$ does exist.

$A_{11} = -1-8 = -9$

$A_{12} = -(-2-6) = 8$

$A_{13} = -8+3 = -5$

$A_{21} = -(-0+8) = -8 \quad A_{22} = 1+6 = 7$ [1]

$A_{23} = -(-4+0) = -4$

$A_{31} = 0-2 = -2$

$A_{32} = -(2-4) = 2$

$A_{33} = -1-0 = -1$

$adj(A')=\begin{bmatrix}-9 & -8 & -2\\ 8 & 7 & 2\\ -5 & -4 & -1\end{bmatrix}$ [½]

$(A')^{-1} = \frac{1}{|A|}adj(A')$

$= \begin{bmatrix}-9 & -8 & -2\\ 8 & 7 & 2\\ -5 & -4 & -1\end{bmatrix}$ [1]

7. $A^2 = \begin{pmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{pmatrix}\begin{pmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{pmatrix} = \begin{pmatrix}5 & -1 & 2\\ 9 & -2 & 5\\ 0 & -1 & -2\end{pmatrix}$ [1]

$A^2 - 5A + 16I$

$= \begin{pmatrix}5 & -1 & 2\\ 9 & -2 & 5\\ 0 & -1 & -2\end{pmatrix}-\begin{pmatrix}10 & 0 & 5\\ 10 & 5 & 15\\ 5 & -5 & 0\end{pmatrix}+\begin{pmatrix}16 & 0 & 0\\ 0 & 16 & 0\\ 0 & 0 & 16\end{pmatrix}$ [2]

$= \begin{pmatrix}11 & -1 & -3\\ -1 & 9 & -10\\ -5 & 4 & 14\end{pmatrix}$ [1]

8. $A^3 = A^2.A = \begin{bmatrix}5 & 0 & 8\\ 2 & 4 & 5\\ 8 & 0 & 13\end{bmatrix}\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}$

$= \begin{bmatrix}5+0+16 & 0+0+0 & 10+0+24\\ 2+0+10 & 0+8+0 & 4+4+15\\ 8+0+26 & 0+0+0 & 16+0+39\end{bmatrix}$

$= \begin{bmatrix}21 & 0 & 34\\ 12 & 8 & 23\\ 34 & 0 & 55\end{bmatrix}$ [1]

Taking $A^3 - 6A^2 + 7A + kI_3 = 0$

$$\begin{bmatrix}21 & 0 & 34\\12 & 8 & 23\\34 & 0 & 55\end{bmatrix} - 6\begin{bmatrix}5 & 0 & 8\\2 & 4 & 5\\8 & 0 & 13\end{bmatrix} + 7\begin{bmatrix}1 & 0 & 2\\0 & 2 & 1\\2 & 0 & 3\end{bmatrix} + kI_3 = 0$$

[1]

$$\text{Or}\begin{bmatrix}21 & 0 & 34\\12 & 8 & 23\\34 & 0 & 55\end{bmatrix} - \begin{bmatrix}30 & 0 & 48\\12 & 24 & 30\\48 & 0 & 78\end{bmatrix} +$$

$$\begin{bmatrix}7 & 0 & 14\\0 & 14 & 7\\14 & 0 & 21\end{bmatrix} + kI_3 = 0$$

[1]

$$\text{Or}\begin{bmatrix}-2 & 0 & 0\\0 & -2 & 0\\0 & 0 & -2\end{bmatrix} + \begin{bmatrix}k & 0 & 0\\0 & k & 0\\0 & 0 & k\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

$\Rightarrow k - 2 = 0$ [1]

$\Rightarrow k = 2$

9. Given write in matrix form

AX = B [1]

$$\begin{bmatrix}1 & -1 & 2\\3 & 4 & -5\\2 & -1 & 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}7\\-15\\12\end{bmatrix}$$

$AX + B$

$\Rightarrow A^{-1}(AX) = A^{-1}B$

$X = A^{-1}B$

$|A| = 1(12-5) + 1(9+10) + 2(-3-8)$ [1]

$= 7 + 19 - 22 = 4 \neq 0$

A^{-1}exist

$A_{11} = 12 - 5 = 7,$ $A_{12} = -(9 + 10) = -9$

$A_{13} = -3 - 8 = -11,$ $A_{21} = -(-3 + 2) = 1$

$A_{22} = 3 - 4 = -1,$ $A_{23} = -(-1 + 2) = -1$

$A_{31} = 5 - 8 = -3,$ $A_{32} = -(-5 - 6) = 11$

$A_{33} = 4 + 3 = 7$

$$adj.A = \begin{bmatrix}7 & 1 & -3\\-19 & -1 & 11\\-11 & -1 & 7\end{bmatrix}$$ [1½]

$$A^{-1} = \frac{1}{|A|}adjA$$

$$= \frac{1}{4}\begin{bmatrix}7 & 1 & -3\\-19 & -1 & 11\\-11 & -1 & 7\end{bmatrix}$$ [½]

Form:(i) .x = $A^{-1}B$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \frac{1}{4}\begin{bmatrix}7 & 1 & -3\\-19 & -1 & 11\\-11 & -1 & 7\end{bmatrix}\begin{bmatrix}7\\-5\\12\end{bmatrix}$$ [1]

$$= \frac{1}{4}\begin{bmatrix}49-5-36\\-133+5+132\\-77+5+84\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix}8\\4\\12\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\1\\3\end{bmatrix}$$

$x = 2, y = 1, z = 3$ [1]

10. *Let* $A = \begin{bmatrix}x & y\\a & b\end{bmatrix}$ [1]

Therefore

$$\begin{bmatrix}2 & -1\\1 & 0\\-3 & 4\end{bmatrix}\begin{bmatrix}x & y\\a & b\end{bmatrix} = \begin{bmatrix}-1 & -8\\1 & -2\\9 & 22\end{bmatrix}$$ [1]

$$\begin{bmatrix}2x-a & 2y-b\\x & y\\-3x+a & 4y+b\end{bmatrix} = \begin{bmatrix}-1 & -8\\1 & -2\\9 & 22\end{bmatrix}$$ [1]

On equating the L.H.S with R.H.S, we get,

$2x - a = -1$

$2y - b = -8$

$x = 1$

$y = -2$ [1]

Substituting the values of x and y in above equations we get,

$2(1) - a = -1$

$a = 3$

$2(-2) - b = -8$ [1]

So, $b = 4$

From the equations above we get,

$x = 1,\ y = -2,\ a = 3,\ b = 4$

$$A = \begin{bmatrix}1 & -2\\3 & 4\end{bmatrix}$$ [1]

11. Given equations are:

$x - y + z = 4$

$2x + y - 3z = 0$

$x + y + z = 2$

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$AX = B$ [1]

$X = A^{-1}B$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

$|A| = 1(1+3) + 1(2+3) + 1(2-1)$

$= 4 + 5 + 1$

$= 10$ [1]

Now,

$A_{11} = 4$

$A_{12} = -5$

$A_{13} = 1$

$A_{21} = 2$

$A_{22} = 0$

$A_{23} = -2$ [1]

$A_{31} = 2$

$A_{32} = 5$

$A_{33} = 3$

$$adjA = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$ [1]

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 16+4 \\ -20+10 \\ 4+6 \end{bmatrix}$$ [1]

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$x = 2, y = -1, z = 1$ [1]

12. $$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$ [1]

$$= \begin{bmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ 0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4+4 & 3+6-8 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Thus, $AB = I$

Multiplying both sides by P^{-1}

$A^{-1}AB = A^{-1}I$

$IB = P^{-1}$

$B = A^{-1}$ [1]

Given:

$x + 3z = 9, -x + 2y - 2z = 4, \ 2x - 3y + 4z = -3.$

Writing the equations in matrix form we get

$$\begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & -2 \\ 2 & -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix}$$ [1]

$A^T X = C$

Pre multiplying by inverse of A^T both sides

$$\left(A^T\right)^{-1} A^T X = \left(A^T\right)^{-1} C$$

$$IX = \left(A^T\right)^{-1} C$$

$$X = \left(A^T\right)^{-1} C$$

$$X = B^T C$$ [1]

[Hint: $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T = B^T$ [From (1) we get]

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 9 & 6 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix}$$ [1]

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -18+36-18 \\ 0+8-3 \\ 9-12+6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

Thus, $x = 0,\ y = 5,\ z = 3$ [1]

13. Given equations can be written as

$$AX = B$$

$$\Rightarrow X = A^{-1}B$$ [1]

Where $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$, $X = \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

$$|A| = 2(120-45) - 3(-80-30) + 10(36+36)$$

$$= 2(75) - 3(-110) + 10(72)$$

$$= 150 + 330 + 720 = 1200 \neq 0$$ [1]

A^{-1} exists

$A_{11} = 120 - 45 = 75$, $A_{12} = -(-80-30) = 110$

$A_{13} = 36 + 36 = 72$, $_{21} = -(-60-90) = 150$

$A_{22} = -40 - 60 = -100$ $A_{23} = -(18-18) = 0$

$A_{31} = 15 + 60 = 75$, $A_{32} = -(10-40) = 30$,

$A_{33} = -12 - 12 = -24$

$$adj\,(A) = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$ [1]

$$A^{-1} = \frac{1}{|A|} adj(A)$$

$$= \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$ [1]

Form (i) $X = A^{-1}B$

$$\begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{1200} \begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix}$$

$$= \frac{1}{1200} \begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$$ [1]

$$\begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

$$\frac{1}{X} = \frac{1}{2}, \frac{1}{Y} = \frac{1}{3}, \frac{1}{Z} = \frac{1}{5}$$

$\therefore\quad X = 2, Y = 3, Z = 5$ [1]

14. Given $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$$|A| = 2\begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} + 3\begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} + 5\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}$$

$$|A| = 2(-4+4) + 3(-6+4) + 5(3-2)$$

$$|A| = -6 + 5 = -1 \neq 0$$

Thus A^{-1} exist. [1]

Finding cofactors.

$A_{11} = -4+4 = 0,$

$A_{21} = -1(6-5) = -1,$

$A_{31} = 12-10 = 2,$

$A_{12} = -1(-6+4) = 2, \quad A_{13} = 3-2 = 1$
$A_{22} = -4-5 = -9, \quad A_{23} = -1(2+3) = -5$
$A_{32} = -1(-8-15) = 23, \quad A_{33} = 4+9 = 13$

$$\text{Adj } A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$ [1]

Formula , $A^{-1} = \frac{1}{|A|}$ Adj A

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} \quad ...(i)$$

$$A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$ [1]

Now considering $\begin{aligned} 2x-3y+5z &= 11 \\ 3x+2y-4z &= -5 \\ x+y-2z &= -3 \end{aligned}$

Writing the above equation in matrix form.

$AX = B$

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -5 \\ -3 \end{pmatrix}$$ [1]

$AX = B$

$\Rightarrow X = A^{-1}B$

From above equation (i) we get

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{pmatrix} 11 \\ -5 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix}$$ [1]

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow x = 1, y = 2, z = 3$ [1]

15. Product of the matrices

$$\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix}$$ [1]

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8I_3$$

Hence $\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$ [1]

Now, given system of equations can be written in matrix form, as follows

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}^{-1}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$ [1]

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}^{-1} \times \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$ [1]

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \times \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$ [1]

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix}$$

$\therefore \quad x = \frac{24}{8}, y = \frac{-16}{8}, z = \frac{-8}{8}$

$\therefore \quad x = 3, y = -2, z = -1$ [1]

Value Based Questions

PREVIOUS YEARS'

EXAMINATION QUESTIONS

4 Mark Questions

1. The monthly incomes of Aryan and Babban are in the ratio 3 : 4 and their monthly expenditures are in the ratio 5 : 7. If each saves 15,000 per month, find their monthly incomes using matrix method. This problem reflects which value?

[ALL INDIA 2017]

2. Two schools A and B want to award their selected students on the values of sincerity, truthfulness and help-fullness. The school A wants to award Rs. x each, Rs. y each and Rs. z each for the three respective values to 3, 2 and 1 students respectively with a total award money of Rs 1,600. School B wants to spend Rs 2,300 to award its 4, 1 and 3 students on the respective values (by giving the same award money to the three values as before). If the total amount of award for one prize on each value is Rs 900, using matrices, find the award money for each value. Apart from these three values, suggest one more value which should be considered forward.

[ALL INDIA 2014]

3. A typist charges Rs 145 for typing 10 English and 3 Hindi pages, while charges for typing 3 English and 10 Hindi pages are Rupees 180. Using matrices, find the charges of typing one English and one Hindi page separately. However typist charged only Rs 2 per page from a poor student Shyam for 5 Hindi pages. How much less was charged from this poor boy? Which values are reflected in this problem?

[ALL INDIA 2016]

4. The management committee of a residential colony seceded to award some of it's members (say x) for honesty, some (say y) for helping others and some other (say z) for supervising the workers to keep the colony neat and clean. The sum of all the awardees is 12. Three times the sum of awardees for cooperation and supervision added to two times the number of awardees for honesty is 33. If the sum of the numbers of awardees for honesty and supervision is twice the number of awardees for helping others, using matrix method, find the number of awardees of each category. Apart from these values, namely, honesty, cooperation and supervision, suggest one more value which the management of the colony must include for awards.

[DELHI 2013]

5. Two schools p and q want to award the selected students on the value of discipline, politeness and punctuality. The school P wants to award Rs x each, Rs y each and Rs z each for the three respective values to its 3, 2 and 1 students with the total award money of Rs 1,000. School Q wants to spend Rs 1,500 to award its 4, one and Three Students on the respective values (by giving the same award money for the three values as before). If the total amount of awards for one prize on each value is Rs 600, using matrices, find the award money for each value. Apart from the above values, suggest one more value for awards.

[DELHI 2014]

6. A School wants to award its student for the values of honesty , Regularity and Hardwork with a total cash award of rs 6,000. Three times the award money for hardwork added to that given for honesty amounts to rs 11,000 The award money given for Honesty and hardwork together is double the one given for regularity ?

   Represent the above situation algebraically and find the award money for each value ,using matrix method.

[DELHI 2013]

Solutions

1. The monthly incomes of Aryan and Babban are **$3x$ and $4x$** respectively.

Monthly expenditure of Aryan and Babban are **$5y$ and $7y$** respectively.

$\therefore\ 3x - 5y = 15,000$

$4x - 7y = 15,000$ [½]

Writing in matrix form

$$\begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15,000 \\ 15,000 \end{bmatrix}$$

$A.X = B$

$A^{-1}(AX) = A^{-1}B$ [Pre-multiplying by A^{-1}]

$IX = A^{-1}B$

$X = A^{-1}B$ .....(*i*)

$|A| = -21 + 20 = -1 \neq 0$

$\because\ A^{-1}$ does exist

$$adj\,A = \begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \times adj\,A$$

$$= \frac{1}{-1}\begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}$$ [½]

From (i),

$X = A^{-1}B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 15,000 \\ 15,000 \end{bmatrix}$$

$$= \begin{bmatrix} 105,000 - 75,000 \\ 60,000 - 45,000 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30,000 \\ 15,000 \end{bmatrix}$$ [1]

$\therefore\ x = 30,000\ ,\ y = 15,000$

The monthly incomes of Aryan $= 3x = 90,000$ [1]

The monthly incomes of Babban $= 4x = 120,000$

This problem indicates that people should save for secure future. [1]

2. Let Matrix D represents number of students receiving prize for the three categories :

Number of Students of School	Sincerity	Thruthfulness	Helpfulness
A	3	2	1
B	4	1	3
One student of each value.	1	1	1

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where *x*, *y* and *z* are rupees mentioned as it is for sincerity, truthfulness and helpfulness respectively.

$E = \begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix}$ is a matrix representing total award money for school *A*, *B* and for one prize for each value.

We can represent the given question in matrix multiplication as :

$DX = E$

Or

$$\begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix}$$

Solution of the matrix equation exist if $|D| \neq 0$

i.e., $\begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 3[1-3] - 2[4-3] + 1[4-1]$

$= -6 - 2 + 3$

$= -5$

Therefore, the solution of the matrix equation is

$X = D^{-1}E$

To find D^{-1}; $D^{-1} = \frac{1}{|D|}\text{adj}(D)$

Cofactor Matrix of D

$$= \begin{bmatrix} -2 & -1 & 3 \\ -1 & 2 & -1 \\ 5 & -5 & -5 \end{bmatrix}$$

Adjoint of D = adj (D)

$$= \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 5 & -1 & -5 \end{bmatrix}$$

{transpose of Cofactor Matrix}

$$D^{-1} = \frac{1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}\begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix}$$

$\therefore$ $x = 200$, $y = 300$, $z = 400$.

Award can also be given for Punctuality.

3. Charges of typing one English page = Rs x

Charges of typing one Hindi page = Rs y

$10x + 3y = 145$

$3x + 10y = 180$

$Ax = B$

$A^{-1}Ax = A^{-1}B$

$Ix = A^{-1}B$

Now $A = \begin{bmatrix} 10 & 3 \\ 3 & 10 \end{bmatrix}$

$|A| = 100 - 9 = 91$

and $x = \frac{\text{adj}A}{|A|}B$

$$x = \frac{1}{91}\begin{bmatrix} 10 & 3 \\ 3 & 10 \end{bmatrix}\begin{bmatrix} 145 \\ 180 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{91}\begin{bmatrix} 1450 - 540 \\ -435 + 1800 \end{bmatrix} = \begin{bmatrix} 910 \\ 1365 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$$

$x = 10$

$y = 15$

From poor student, he charged = 2 × 5 = Rs 10

Actual cost = 15 × 5 = Rs 75

Less charged = 75 – 10 = Rs 65

This problem reflects, human values of 'kindness'.

4. We know

x + y + z = 12 ...(i)

3(y + z) + 2x = 33 ...(ii)

(x + z) = 2y ...(iii)

$x + y + z = 12$

$2x + 3y + 3z = 33$

$x - 2y + z = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}$$

$AX = B$ [1]

$A^{-1}(AX) = A^{-1}(B)$

$I.X = A^{-1}(B)$

$$X = \frac{(Adj.A)B}{|A|}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{vmatrix}$$

$|A| = 1(3+6) - 1(2-3) + 1(-4-3)$

$|A| = 9 + 1 - 7 = 3$ [1]

$|A| \neq 0$

$$\text{Adj}A = \begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$$

$$(\text{Adj}A).B = \begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}_{(3\times3)} \begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}_{(3\times1)}$$

$$(\text{Adj}A).B = \begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix}$$ [1]

$$\therefore X = \frac{(Adj.A).B}{|A|}$$

$$X = \frac{1}{3}\begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

$\therefore \; x = 3, y = 4, z = 5.$ [1]

5. $$\begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 600 \end{bmatrix}$$ [1]

$AX = B$

$A^{-1}(AX) = A^{-1}B$

$X = A^{-1}B$ ...(i)

$|A| = 3(1-3) - 2(4-3) + 1(4-1)$

$= -6 - 2 + 3 = -5 \neq 0$

A^{-1} exists

$A_{11} = 1 - 3 = -2$

$A_{12} = -(4-3) = -1$

$A_{13} = 4 - 1 = 3$

$A_{21} = -(2-1) = -1$

$A_{22} = 3 - 1 = 2$

$A_{23} = -(3-2) = -1$

$A_{31} = 6 - 1 = 5$

$A_{31} = 6 - 1 = 5$

$A_{31} = 6 - 1 = 5$

$$\therefore \; adj\,A = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$ [1]

$$\therefore a^{-1} = \frac{1}{|A|}.\,adj\,A$$

$$= -\frac{1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

from (i), $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}\begin{bmatrix} 1000 \\ 1500 \\ 600 \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} -2000 - 1500 + 3000 \\ -1000 + 3000 - 3000 \\ 3000 - 1500 - 3000 \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} -500 \\ -1000 \\ -1500 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$$ [1]

$\therefore x = 100, y = 200, z = 300$ is the award money for each value.

Other values for awards may be: honesty, hard work, regularity, obedience etc. [1]

6. Given :Let the award money for honesty = Rs. x,
The award money for regularity = Rs. y
And the award money for Hardwork = Rs. z
According to the question ,

$x + y + z = \text{Rs.}6{,}000$

$x + 3z = \text{Rs.}11{,}000$

$x + z = 2(y)$

$x - 2y + z = 0$ [1]

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix}}\underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 6{,}000 \\ 11{,}000 \\ 0 \end{bmatrix}}$$

Writing in matrix from [1]

$$A^{-1}(AX) = A^{-1}B \text{....Where} \left[B = \begin{bmatrix} 6,000 \\ 11,000 \\ 0 \end{bmatrix} \right]$$

$$X = A^{-1}B$$

$$|A| = 1(0+6) - 1(1-3) + 1(-2-0)$$

$$= 6 + 2 - 2 = 6 \neq 0$$

$A_{11} = 0 + 6 = 6$, $A_{12} = -(-2) = 2$

$A_{13} = -2$, $A_{21} = -(3) = -3$

$A_{22} = 1 - 1 = 0$ $A_{23} = -(-3) = 3$

$A_{31} = 3 - 0 = 3$, $A_{32} = -(2) = -2$

$A_{33} = -1$

A^{-1} exists as A is non –singular

$$\therefore \ adjA = \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

Now, $A^{-1} = \frac{1}{|A|} adj\,A$

$$= \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$ [1]

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 36000 - 33000 + 0 \\ 12000 + 0 - 0 \\ -12000 + 33000 - 0 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 3000 \\ 12000 \\ 21000 \end{bmatrix} = \begin{bmatrix} 500 \\ 2000 \\ 3500 \end{bmatrix}$$

x = Rs. 500, y = 2,000, z = 3,500

Award money for honestly x = Rs. 500

Award money for regularity, y = Rs. 2,000

Award money for hardwork, z = Rs. 3,500 [1]

Smart Notes ....

Smart Notes ....

CHAPTER 5

Continuity and Differentiability

Chapter Analysis of Previous Three Years' Board Exams

Number of Questions Asked in Exams

Topics	2018		2017		2016	
	Delhi	**All India**	**Delhi**	**All India**	**Delhi**	**All India**
Continuity of a Function			1 mark	1 mark	4 marks	
Differentiation Based on Formula	2, 4, 4 marks	2, 4, 4 marks	2 marks	4 marks		4 marks
Differentiation at a Point					4 marks	
Logarithmic Differentiation			4 marks			4 marks

[TOPIC 1] Continuity

Summary

- **Definition of Continuity:**

  Let f is a real valued function and is a subset of real numbers and a point c lies in the domain of f, then f is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

  When the function f is discontinuous at c, it is called the point of discontinuity of f.

  Also, if f is defined on $[a, b]$ then continuity of a function f at a means

$$\lim_{x \to a^+} f(x) = f(a)$$

  And continuity of the function f at b means

$$\lim_{x \to b^-} f(x) = f(b)$$

- Every polynomial function is continuous.
- Consider two real functions f and g which are continuous at c, then sum, difference, product and quotient of the two functions will also be continuous at $x = c$.

  i.e. $(f + g)(x) = f(x) + g(x)$ is continuous at $x = c$

  $(f - g)(x) = f(x) - g(x)$ is continuous at $x = c$

  $(f \,.\, g)(x) = f(x) \,.\, g(x)$ is continuous at $x = c$

  Here if f is a constant function say $f(x) = \alpha$ for some real number α, then the function $(\alpha \,.\, g)$ defined by $(\alpha \,.\, g)(x) = \alpha \,.\, g(x)$ is also continuous. If $\alpha = -1$ then continuity of f implies continuity of $-f$.

  $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ is continuous at $x = c$ when $g(x) \neq 0$

  Here, if f is a constant function say $f(x) = \alpha$ for some real number α, then the function $\frac{\alpha}{g}$ defined by

  $\frac{\alpha}{g}(x) = \frac{\alpha}{g(x)}$ is also continuous wherever $g(x) \neq 0$.

- **Here are some formulae for limits:**

  - $\lim_{x \to 0} \cos x = 1$
  - $\lim_{x \to 0} \frac{\sin x}{x} = 1$
  - $\lim_{x \to 0} \frac{\tan x}{x} = 1$
  - $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1$
  - $\lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1$
  - $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a,\ a > 0$
  - $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$
  - $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$
  - $\lim_{x \to 0} \frac{\log_e(1 + x)}{x} = 1$
  - $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

PREVIOUS YEARS' EXAMINATION QUESTIONS

TOPIC 1

1 Mark Questions

1. Determine the value of the constant '*k*' so that the

function $(x)=\begin{cases}\dfrac{kx}{|x|}, \text{ if } x<0\\ 3, \text{ if } x\geq 0\end{cases}$ continuous at $x=0$.

[DELHI 2017]

2. Determine the value of '*k*' for which the following function is continuous at $x=3$.

$$f(x)=\begin{cases}\dfrac{(x+3)^2-36}{x-3}, x\neq 3\\ kx, x=3\end{cases}$$

[ALL INDIA 2017]

4 Marks Questions

3. Find the values of *p* and *q*, for which

$$f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x}, \quad \text{if } x<\dfrac{\pi}{2}\\ p, \quad \text{If } x=\dfrac{\pi}{2} \text{ is continuous at } x=\dfrac{\pi}{2}\\ \dfrac{q(1-\sin x)}{(\pi-2x)^2}, \text{if } x>\dfrac{\pi}{2}\end{cases}$$

[DELHI 2016]

4. Find the value of '*a*' for which the function *f*

defined as $f(x)=\begin{cases}a\sin\dfrac{\pi}{2}(x+1,) & x\leq 0\\ \dfrac{\tan x-\sin x}{x^3}, & x>0\end{cases}$

[DELHI 2011]

5. Find the value of k so that the function *f* defined by

$$f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, \text{if } x\neq\dfrac{\pi}{2}\\ 3, \text{if } x=\dfrac{\pi}{2}\end{cases} \text{ is continuous at } x=\frac{\pi}{2}$$

[ALL INDIA 2011]

6. Find the value of *k*, for which

$$f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, \text{if } -1\leq x<0\\ \dfrac{2x+1}{x-1}, \text{if} 0\leq x<1\end{cases}$$

is continuous at $x=0$

[ALL INDIA 2013]

7 If $f(x)=\begin{cases}\sin(a+1)x+2\sin x, x<0\\ 2, x=0\\ \dfrac{\sqrt{1+bx}-1}{x}, x>0\end{cases}$

is continuous at $x=0$, then find the values of *a* and *b*.

[ALL INDIA 2016]

Solutions

1. Given $(x)=\begin{cases}\dfrac{kx}{|x|}, \text{ if } x<0\\ 3, \text{ if } x\geq 0\end{cases}$

For mode function $\begin{matrix}|x|=-x, \ x<0\\ |x|=x, \ x\geq 0\end{matrix}$

Left hand limit

$\lim\limits_{x\to 0^-} f(x)$

$=\lim\limits_{x\to 0^-}\dfrac{kx}{|x|}$

$=\lim\limits_{x\to 0^-}\dfrac{kx}{-x}$ [½]

$=\lim\limits_{x\to 0^-}-k$

$=-k$

Right hand limit

$\lim\limits_{x\to 0^+} f(x)$

$=\lim\limits_{x\to 0^+} 3$

$=3$

At the point Limit

$\lim_{x\to 0} f(x)$

$= \lim_{x\to 0} 3$

$= 3$

Given that f is continuous at $x = 0$.

$-k = 3 = 3$

$\Rightarrow k = -3$ [½]

2. $\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{(x+3)^2 - 36}{x-3}$

$= \lim_{x\to 3} \frac{(x+3-6)(x+3+6)}{x-3}$

$= 12$

given that f(x) is continuous at $x = 3$

$f(x) = \lim_{x\to 3} kx$

$= 3k$

$\Rightarrow 3k = 12$

$k = 4$ [1]

3. LHS

$\lim_{x\to \frac{\pi^-}{2}} f(x) = \lim_{x\to \frac{\pi^-}{2}} \frac{1-\sin^3 x}{3\cos^2 x}$ [1]

$= \lim_{x\to \frac{\pi^-}{2}} \frac{(1-\sin x)(1+\sin^2 x+\sin x)}{3(1-\sin^2 x)}$

$\left[\because a^3 - b^3 = (a-b)(a^2+ab+b^2)\right)$

$= \lim_{x\to \frac{\pi^-}{2}} \frac{(1-\sin x)(1+\sin^2 x+\sin x)}{3(1-\sin x)(1+\sin x)}$

$= \frac{1+\sin^2\frac{\pi}{2}+\sin\frac{\pi}{2}}{3\left(1+\sin\frac{\pi}{2}\right)} = \frac{1+1+1}{3(1+1)} = \frac{3}{6} = \frac{1}{2}$ [1]

RHL $= \lim_{x\to \frac{\pi^+}{2}} \frac{q(1-\sin x)}{(\pi-2x)^2}$

$= \lim_{x\to \frac{\pi^-}{2}} \frac{(1-\sin x)(1+\sin^2 x+\sin x)}{3(1-\sin^2 x)}$ [1]

$\left[\because a^3 - b^3 = (a-b)(a^2+ab+b^2)\right)$

$= \lim_{t\to 0} \frac{q\left[1-\sin\left(\frac{\pi}{2}-t\right)\right]}{\left[\pi-2\left(\frac{\pi}{2}-t\right)\right]^2}$

Let $t = \frac{\pi}{2} - x$

$x = \frac{\pi}{2} - t$

As $x \to \frac{\pi}{2}$,

$t \to 0$

$= \lim_{t\to 0} \frac{q(1-\cos t)}{[\pi-\pi+2t]^2}$

$= \lim_{t\to 0} \frac{q\left(2\sin^2\frac{t}{2}\right)}{4\times 4\times\frac{t^2}{4}} = \frac{2q}{16}\lim_{t\to 0}\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2$

$= \frac{q}{8}(1)$ $\left(\because \lim_{t\to 0}\left(\frac{\sin\theta}{\theta} = 1\right)\right)$

$= \frac{q}{8}$

At $x = \frac{\pi}{2}, f(x) = p \Rightarrow f\left(\frac{\pi}{2}\right) = p$

$f(x)$ is continuous at $x = \frac{\pi}{2}$ Given

i.e., LHL = RHL $= f\left(\frac{\pi}{2}\right)$

$\frac{1}{2} = \frac{q}{8} = p$

$\therefore \frac{1}{2} = \frac{q}{8}$

$q = 4$

$\therefore$ If $q = 4 \Rightarrow p = \frac{1}{2}$

$\therefore p = \frac{1}{2}, q = 4$ [1]

4. $\lim_{x\to 0} f(x)$

$= \lim_{x\to 0} a \sin\left[\frac{\pi}{2}(x+1)\right]$

$= \lim_{x\to 0} a \sin\left[\frac{\pi}{2}(0-h+1)\right]$

$= a \sin\left[\frac{\pi}{2}(0-0+1)\right] = a\sin\frac{\pi}{2}$

$= a(1) = a$ [1]

R. H. L. = $\lim_{x\to 0} f(x)$

$= \lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$

Using : $\left[\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right]$ and $\left[\lim_{\theta\to 0} \frac{\tan\theta}{\theta} = 1\right]$ [1]

$\lim_{h\to 0} \frac{\tan(0+h) - \sin(0+h)}{(0+h^3)}$

$= \lim_{h\to 0} \frac{\tan h - \sin h}{h^3}$

$= \lim \frac{\frac{\sin h}{\cos h} - \frac{\sin h}{1}}{h^3}$

$= \lim \frac{\sin h - \sin h \cos h}{\cos h} \times \frac{1}{h^3}$ [1]

$= \lim_{h\to 0} \frac{\sin h(1-\cos h)}{\cos h . h^3}$

$= \lim_{h\to 0} \frac{\tan h\left(2\sin^2 \frac{h}{2}\right)}{h.4\frac{h^2}{4}}$

$= \frac{2}{4} \lim_{h\to 0}\left(\frac{\tan h}{h}\right) . \lim_{h\to 0}\left(\frac{\sin \frac{h}{2}}{\frac{h}{2}}\right)^2$

$= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

$a = \frac{1}{2}$

At $x = 0, f(x) = a \sin\left[\frac{\pi}{2}(x+1)\right]$

$f(o) = a \sin\left[\frac{\pi}{2}(0+1)\right]$

$= a \sin\frac{\pi}{2} = a(1) = a$

$f(x)$ is continuous at $x = 0$

L.H.L. = R.H.L. = $f(0)$

$a = \frac{1}{2}$ [1]

5. $f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, \text{if } x \neq \frac{\pi}{2} \\ 3, \text{if } x = \frac{\pi}{2} \end{cases}$ at $x = \frac{\pi}{2}$

$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \left(\frac{k\cos x}{\pi - 2x}\right)$ [1]

Put $x = \frac{\pi}{2} + h$

$= \lim_{h\to 0}\left[\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi - 2\left(\frac{\pi}{2}+h\right)}\right]$ [1]

$= \lim_{h\to 0}\left[\frac{-\sin h}{-2h}\right]$

$= \frac{k}{2} \lim_{h\to 0}\left[\frac{\sin h}{h}\right]$

$= \frac{k}{2} . 1$

$= \frac{k}{2}$ [1]

Thus,

$$\lim_{x\to\frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\therefore \frac{k}{2} = 3$$

$k = 6$ [1]

6. Given equations can be written as

$$f(x) = \begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, \text{if } -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, \text{if} 0 \le x < 1 \end{cases}$$

Function f is continuous at $x = 0$

$\therefore f(0) = \lim_{x\to 0} f(x)$ [1]

$$\therefore \frac{0+1}{0-1} = \lim_{x\to 0}\left(\frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}\right)$$

$$\therefore -1 = \lim_{x\to 0}\left(\frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}\right)\left(\frac{\sqrt{1+kx}+\sqrt{1-kx}}{\sqrt{1+kx}+\sqrt{1-kx}}\right)$$

$[\because$ Rationalize$]$ [1]

$$\therefore \quad -1 = \lim_{x\to 0}\frac{(1+kx)-(1-kx)}{x[\sqrt{1+kx}+\sqrt{1-kx}]}$$

$\therefore -1 = \lim_{x\to 0}\dfrac{2kx}{x\left(\sqrt{1+kx}+\sqrt{1-kx}\right)}$ [1]

$$\therefore -1 = \lim_{x\to 0}\frac{2k}{\sqrt{1+kx}+\sqrt{1-kx}}$$

$$\therefore -1 = \frac{2k}{2}$$

$\therefore \quad k = -1$ [1]

7. $f(x) = \begin{cases} \sin(a+1)x + 2\sin x, x < 0 \\ 2, x = 0 \\ \dfrac{\sqrt{1+bx}-1}{x}, x > 0 \end{cases}$

Value at $x = 0$ is 2

LHL: $\lim_{h\to 0} f(0-h)$ [1]

$$= \lim_{h\to 0}\frac{-\sin(a+1)h - 2\sin h}{-h}$$

$= \lim_{h\to 0}\left(\dfrac{\sin(a+1)h}{h} + \dfrac{2\sin h}{h}\right)$ [1]

$$\left(\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\right)$$

$$(a+1)+2 = a+3$$

RHL: $\lim_{h\to 0} f(0+h)$

$$= \lim_{h\to 0}\frac{\left(\sqrt{1+bh}-1\right)}{h}\times\frac{\left(\sqrt{1+bh}+1\right)}{\left(\sqrt{1+bh}+1\right)}$$

$= \dfrac{b}{2}$(rationalise) [1]

Since it is continuous

$\therefore$ L.H.L = R.H.L = $f(0)$

$$\therefore a+3 = \frac{b}{2} = 2$$

$\therefore a = -1, b = 4$ [1]

[TOPIC 2] Differentiability

Summary

- **Differentiability:**

  Consider a real function f and a point c lies in its domain then the derivative of that function at c is defined by

  $$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

  Provided the limit exists. It is denoted by $f'(c)$ or $\frac{d}{dx}(f(x))$.

- **Some rules for algebra of derivatives:**
  - ➢ $(u+v)' = u' + v'$
  - ➢ $(u-v)' = u' - v$
  - ➢ Product rule: $(uv)'\ u'v + uv'$
  - ➢ Quotient rule: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, $v \neq 0$

- A function which is differentiable at a point c is also continuous at that point but the converse is not true.

- **Chain Rule:**

  Consider a real value function f which is a composite of u and v.

  Let $t = u(x)$ and $\frac{dt}{dx}, \frac{dv}{dt}$ exists, then $\frac{df}{dx} = \frac{dv}{dt}.\frac{dt}{dx}$

- **Some important features of exponential function and logarithm function are given below**
  - ➢ Domain of both the functions is set of all real numbers.
  - ➢ Range of exponential function is set of all positive real numbers and the range of log function is set of all real numbers.
  - ➢ The point (0, 1) is always on the graph of exponential function and the point (1, 0) is always on the graph of log function.
  - ➢ Both the functions are ever increasing.

- A relation expressed between two variable x and y in the form $x = f(t)$, $y = g(t)$ is said to be **parametric form** with t as a parameter.

  By using Chain Rule we find the derivative of function in such form.

  $$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$

  or $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $\left(\text{whenever } \frac{dx}{dt} \neq 0\right)$

  Thus $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$

- **Second Order Derivative:**

  If $y = f(x)$

  $$\frac{dy}{dx} = f'(x)$$

  If $f'(x)$ is differentiable then $\frac{dy}{dx} = f'(x)$ will be differentiated again. The left side will become $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ and is called second order derivative of y w.r.t. x.

- **Rolle's Theorem:**

  Consider a real valued function f defined on the interval $[a, b]$ such that the function is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$, then there exist a point c in (a, b) such that $f'(c) = 0$.

- **Lagrange's Mean Value Theorem:**

  Consider a real valued function f defined on the interval $[a, b]$ such that the function is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

- **Derivatives of some standard functions are listed below:**
  - ➢ $\frac{d}{dx}(x^n) = nx^{n-1}$
  - ➢ $\frac{d}{dx}(k) = 0$, k is any constant
  - ➢ $\frac{d}{dx}(a^x) = a^x \log_e a,\ a > 0$
  - ➢ $\frac{d}{dx}(e^x) = e^x$
  - ➢ $\frac{d}{dx}(\log_a x) = \frac{1}{x\log_e a} = \frac{1}{x}\log_a e$

- $\frac{d}{dx}(\log_e x) = \frac{1}{x}$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\cot x) = -\cos ec^2 x$
- $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$
- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \ x \in (-1, 1)$
- $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \ x \in (-1, 1)$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \ x \in \mathbb{R}$
- $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, \ x \in \mathbb{R}$
- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$, where $x \in (-\infty, -1) \cup (1, \infty)$
- $\frac{d}{dx}(\text{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$, where $x \in (-\infty, -1) \cup (1, \infty)$

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 2

2 Marks Question

1. Find $\frac{dy}{dx}$ at $x = 1, \ y = \frac{\pi}{4}$ if $\sin^2 y + \cos xy = K$

[DELHI 2017]

4 Marks Questions

2. If $x^y + y^x = a^b$, then find $\frac{dy}{dx}$

[ALL INDIA 2017]

3. If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), x^2 \le 1,$ then find $\frac{dy}{dx}$

[DELHI 2015]

4. $e^y(x+1) = 1$ then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

[ALL INDIA 2017]

5. If x cos(a + y) = cos y then prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

Hence show that $\sin a \frac{d^2y}{dx^2} + \sin 2(a+y)\frac{dy}{dx} = 0$

[ALL INDIA 2016]

6. **Find** $\frac{dy}{dx}$ **if** $y = \sin^{-1}\left[\frac{6x - 4\sqrt{1-4x^2}}{5}\right]$

[ALL INDIA 2016]

7. Differentiate the following function with respect to x: $(\log x)^x + x^{\log x}$

[DELHI 2013]

8. If y = log $\left[x + \sqrt{x^2 + a^2}\right]$, show that $(x^2 + a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

[DELHI 2013]

9. Show that the function $f(x)=|x-3|, x\in R$ is continuous but not differentiable at $x=3$

[DELHI 2013]

10. If $x=a\sin t$ and $y=a\left(\cos t+\log\tan\frac{t}{2}\right)$, find $\frac{d^2y}{dx^2}$.

[DELHI 2013]

11. Differentiate the function $(\sin x)^x+\sin^{-1}\sqrt{x}$ with respect to x.

[DELHI 2017]

12. If $x^m y^n=(x+y)^{m+n}$, prove that $\frac{d^2y}{dx^2}=0$.

[DELHI 2017]

13. If $x=a\sin 2t(1+\cos 2t)$ and $y=b\cos 2t\,(1-\cos 2t)$, find the values of $\frac{dy}{dx}$ at $t=\frac{\pi}{4}$ and $t=\frac{\pi}{3}$

[DELHI 2016]

14. If $x=a\cos\theta+b\sin\theta, y=a\sin\theta-b\cos\theta$,

show that $y^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0$

[DELHI 2015]

15. If $y=x^x$, prove that $\frac{d^2y}{dx^2}-\frac{1}{y}\left(\frac{dy}{dx}\right)^2-\frac{y}{x}=0$

[DELHI 2016]

16. Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$, when $x\neq 0$

[DELHI 2014]

17. If $y=\left(\tan^{-1}x\right)^2$, show that

$$\left(x^2+1\right)^2\frac{d^2y}{dx^2}+2x\left(x^2+1\right)\frac{dy}{dx}=2.$$

[DELHI 2012]

18. If $\sin y=x\sin(a+y)$, prove that $\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$

[DELHI 2012]

19. Differentiate $x^{x\cos x}+\frac{x^2+1}{x^2-1}$ *w.r.t.* x.

[DELHI 2011]

20. If $x=a(\theta-\sin\theta), y=a(1+\cos\theta)$, *find* $\frac{d^2y}{dx^2}$

[DELHI 2011]

21. If $x=a(2\theta-\sin 2\theta)$ and $y=a(1-\cos 2\theta)$, find $\frac{dy}{dx}$ when $\theta=\frac{\pi}{3}$

[DELHI 2018]

22. If $\left(x^2+y^2\right)^2=xy$ find $\frac{dy}{dx}$

[DELHI 2018]

23. If $y=\sin(\sin x)$, prove that

$$\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x=0$$

[DELHI 2018]

24. Differentiate $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to x.

[DELHI 2018]

25. Differentiate the function with respect to x.

$$y=(x\cos x)^x+(x\sin x)^{\frac{1}{x}}$$

[ALL INDIA 2011]

26. If $y=3\cos(\log x)+4\sin(\log x)$,

show that $x^2y_2+xy_1+y=0$

[DELHI 2011]

27. If $(\cos x)^y=(\cos y)^x$ find $\frac{dy}{dx}$

[ALL INDIA 2012]

28. If $y^x=e^{y-x}$, prove that $\frac{dy}{dx}=\frac{(1+\log y)^2}{\log y}$

[ALL INDIA 2013]

29. Differentiate the following with respect to x:

$$\sin^{-1}\left(\frac{2^{x+1}.3^x}{1+(36)^x}\right)$$

[ALL INDIA 2013]

30. If $x = a\cos^3\theta$ and $y = a\sin^3\theta$ then find the value of $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{6}$

[DELHI 2013]

31. Find the value of

$\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$, if $x = ae^{\theta}(\sin\theta - \cos\theta)$ and

$y = ae^{\theta}(\sin\theta + \cos\theta)$

[ALL INDIA 2014]

32. Find $\frac{d}{dx}\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$

[ALL INDIA 2015]

33. Find the derivative of the following function f(x) w.r.t. x, at x = 1 :

$$\cos^{-1}\left[\sin\sqrt{\frac{1+x}{2}}\right] + x^x$$

[ALL INDIA 2015]

34. If $x = \alpha \sin 2t(1+\cos 2t)$ and

$y = \beta\cos 2t(1-\cos 2t)$, show that $\frac{dy}{dx} = \frac{\beta}{\alpha}\tan t$

[ALL INDIA 2015]

Solutions

1. Given $\sin^2 y + \cos xy = K$

Differentiating both sides w.r.t. x we get

$$2\sin y\cos y\frac{dy}{dx} - \sin xy\left(x\frac{dy}{dx} + y\right) = 0$$

$$\sin 2y\frac{dy}{dx} - x\sin xy\frac{dy}{dx} - y\sin xy = 0$$

$$(\sin 2y - x\sin xy)\frac{dy}{dx} = y\sin xy$$

$$\frac{dy}{dx} = \frac{y\sin xy}{(\sin 2y - x\sin xy)}$$ [1]

$$\left.\frac{dy}{dx}\right|_{\left(1,\frac{\pi}{4}\right)} = \frac{\frac{\pi}{4}\sin\left(\frac{\pi}{4}\right)}{\left(\sin 2\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right)}$$

$$= \frac{\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)}{1-\frac{1}{\sqrt{2}}} = \frac{\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)}{\frac{\sqrt{2}-1}{\sqrt{2}}}$$

$$= \frac{\frac{\pi}{4}}{\sqrt{2}-1}$$

$$= \frac{\pi}{4}\times\frac{1}{(\sqrt{2}-1)}\times\frac{(\sqrt{2}+1)}{(\sqrt{2}+1)}$$

$$= \frac{\pi}{4}\frac{(\sqrt{2}+1)}{(2-1)} = \frac{\pi(\sqrt{2}+1)}{4}$$ [1]

2. Let $u = x^y$ and $v = y^x$

Then, $u + v = a^b$

Differentiation both sides w.r.t. x, we get

$$\frac{du}{dx} + \frac{dv}{dx} = 0 \quad ...(1)$$

Now, $u = x^y$

$\log u = y\log x$ [1]

Differentiating both sides w.r.t. x, we have

$$\frac{1}{u}\frac{du}{dx} = yx\frac{1}{x} + \log x\frac{dy}{dx}$$

$$\frac{du}{dx} = \frac{x^y\left((y + x\log x\frac{dy}{dx}\right)}{x}$$

$$\frac{du}{dx} = x^{y-1}\left(y + x\log\frac{dy}{dx}\right) \quad ...(2)$$ [1]

Now,

$v = y^x$

$\log v = x\log y$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{v}\frac{dv}{dx} = x X \frac{1}{y} X \frac{dy}{dx} + \log y$$

$$\frac{dv}{dx} = y^x \frac{x X \frac{dy}{dx} + ylogy}{y}$$

$$\frac{dv}{dx} = y^{x-1}\left(x\frac{dy}{dx} + ylog\, y\right) \quad ...(3)$$ [1]

From (1),(2) and (3), we have

$$x^{y-1}\left(y + xlog\, x\frac{dy}{dx}\right) + y^{x-1}\left(x\frac{dy}{dx} + ylog\, y\right) = 0$$

$$\frac{dy}{dx} = \frac{y^x \log y - x^{y-1} y}{x^y \log x - xy^{x-1}}$$ [1]

3. $y = \tan^{-1}\left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right)$,

Put $x^2 = \cos 2\theta$

$\cos^{-1} x^2 = 2\theta$

$\frac{1}{2}\cos^{-1} x^2 = \theta$

$$y = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}\right),$$ [1]

$$y = \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta} + \sqrt{2\cos^2\theta}}{\sqrt{2\cos^2\theta} - \sqrt{2\sin^2\theta}}\right),$$

$$y = \tan^{-1}\left(\frac{\sqrt{2}(\cos\theta + \sin\theta)}{\sqrt{2}(\cos\theta - \sin\theta)}\right),$$ [1]

Diividing numerator and denominator by cosθ, we have

$$y = \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right),$$

$$y = \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \theta\right)\right],$$

$$y = \frac{\pi}{4} + \theta$$

$$y = \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$$ [1]

Diffrentiating both sides w.r.t. x, we have,

$$\frac{dy}{dx} = 0 + \frac{1}{2} \times \frac{-1}{\sqrt{1-\left(x^2\right)^2}} \times 2x = \frac{-x}{\sqrt{1-x^4}}$$ [1]

4. The given equation is $e^y(x+1) = 1$

$$x + 1 = \frac{1}{e^y}$$

$$x + 1 = e^{-y}$$ [1]

Differentiating both sides w.r.t. x, we have

$$\frac{dy}{dx} + \frac{d(1)}{dx} = \frac{d\left(e^{-y}\right)}{dx}$$ [1]

$$1 + 0 = -e^{-y}\frac{dy}{dx}$$

$$\frac{dy}{dx} = -e^y$$ [1]

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$ [1]

Hence proved.

5. $x\cos(a+y) = \cos y$

$$x = \frac{\cos y}{\cos(a+y)}$$

Differentiate with respect to ‘x’

$$1 = \frac{\cos(a+y)\left(-\sin y\frac{dy}{dx}\right) - \cos y\left(-\sin(a+y)\frac{dy}{dx}\right)}{\cos^2(a+y)}$$

[1]

$$1 = \frac{\left(\sin(a+y)\cos y - \cos(a+y)\sin y\right)\frac{dy}{dx}}{\cos^2(a+y)}$$

$$\cos^2(a+y) = \sin(a+y-y)\frac{dy}{dx}$$ [1]

$$\therefore \quad \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$ [1]

And

$$\sin a\frac{dy}{dx}=\frac{1+\cos 2(a+y)}{2}$$

Again, Differentiate with respect to 'x'

$$\sin a\frac{d^2y}{dx^2}=0-\frac{\sin 2(a+y)}{2}.2\frac{dy}{dx}$$

$$\sin a\frac{d^2y}{dx^2}+\sin 2(a+y)\frac{dy}{dx}=0 \quad [1]$$

6. $y=\sin^{-1}\left[\frac{6x-4\sqrt{1-x^2}}{5}\right]$

$$=\sin^{-1}\left[\frac{3}{5}.2x-\frac{4}{5}\sqrt{1-4x^2}\right] \quad [1]$$

$$=\sin^{-1}\left[2x.\sqrt{1-\left(\frac{4}{5}\right)^2}-\frac{4}{5}\sqrt{1-(2x)^2}\right] \quad [1]$$

$$y=\sin^{-1}2x-\sin^{-1}\frac{4}{5} \quad [1]$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-(2x)^2}}.2-0=\frac{2}{\sqrt{1-4x^2}} \quad [1]$$

7. Given: Let $y=(\log x)^x+x^{\log x}$

Let $y=A+B$

Differentiating both sides w.r.t. **'x'**, we get

$$\frac{dy}{dx}=\frac{dA}{dx}+\frac{dB}{dx} \quad [1]$$

$$A=(\log x)^x$$

Taking log on both sides

$$\log A=x.\log(\log x)$$

Differentiating both sides w.r.t. 'x', we get

$$\frac{1}{A}.\frac{dA}{dX}=x.\frac{1}{\log x}.\frac{1}{x}+\log(\log x).1$$

$$\frac{dA}{dX}=A\left[\frac{1}{\log x}+\log(\log x)\right] \quad [1]$$

$$B=x^{\log x}$$

Taking log on both sides

$$\log B=\log x.\log x$$

$$=(\log x)^2$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{B}.\frac{dB}{dx}=2\log x.\frac{1}{x}$$

$$\frac{dB}{dx}=B.\left(\frac{2\log x}{x}\right)=x^{\log x}.\frac{2\log x}{x} \quad [1]$$

$$\frac{dy}{dx}=(\log x)^x\left[\frac{1}{\log x}+\log(\log x)\right]+x^{\log x}.\frac{2\log x}{x}$$

[1]

8. Given $y=\log\left[x+\sqrt{x^2+a^2}\right]$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+a^2}}\frac{d}{dx}\left(x+\sqrt{x^2+a^2}\right)$$

$$=\frac{1}{x+\sqrt{x^2+a^2}}\left[1+\frac{1}{2\sqrt{x^2+a^2}}2x\right] \quad [1]$$

$$=\frac{1}{x+\sqrt{x^2+a^2}}\left[\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right]$$

$$=\frac{1}{\sqrt{x^2+a^2}}$$

$$\Rightarrow\sqrt{x^2+a^2}\frac{dy}{dx}=1 \quad [1]$$

Again differentiating both sides w.r.t. x, we get

$$\sqrt{x^2+a^2}.\frac{d^2y}{dx^2}+\frac{dy}{dx}.\frac{1}{2\sqrt{x^2+a^2}}x\,2x=0 \quad [1]$$

$$\frac{(x^2+a^2)\frac{d^2y}{dx^2}+x.\frac{dy}{dx}}{\sqrt{x^2+a^2}}=0$$

$$(x^2+a^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0 \quad [1]$$

9. Given $|x-3| = \begin{cases} -(x-3), & if x < 3 \\ (x-3), & if x \geq 3 \end{cases}$

For continuity at $x = 3$

$L.H.L = \lim_{x\to 3^-} f(x)$ $\quad R.H.L = \lim_{x\to 3^+} f(x)$ [1]

$= \lim_{x\to 3^-} -(x-3)$ $\quad = \lim_{x\to 3^+} (x-3)$

$= \lim_{h\to 0} -[3-h-3]$ $\quad = \lim_{h\to 0} [3+h-3]$

$[put\, x = 3-h, h > 0]$ $\quad [put\, x = 3+h, h > 0]$

$= -3(3-0-3) = 0$ $\quad = 3+0-3 = 0$

$At\, x = 3, f(x) = (x-3) \Rightarrow f(3) = 3-3 = 0$

L.H.L. = R.H.L = $f(3)$

F(x) is continuous at $x = 3$ [1]

For differentiability at $x = 3$

$$L.H.D = \lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3}$$ [1]

$$= \lim_{x\to 3^-} \frac{-(x-3)-0}{(x-3)}$$

$$= \lim_{x\to 3^-} (-1) = -1$$

$$R.H.D. = \lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3}$$

$$= \lim_{x\to 3^+} \frac{(x-3)-0}{(x-3)}$$

$$= \lim_{x\to 3^+} 1 = 1$$

L.H.D ≠ R.H.D.

$f(x)$ is not differentiable at $x = 3$ [1]

10. Given $x = a \sin t$,

$\frac{dx}{dt} = a \cos t$ (Differentiating both sides w.r.t. 't') [1]

$$y = a\left(\cos t + \log \tan \frac{t}{2}\right)$$

$$\frac{dy}{dt} = a\left[-\sin t + \frac{1}{\tan \frac{t}{2}} \times \sec^2 \frac{t}{2} \times \frac{1}{2}\right]$$ [1]

$$= a\left[-\sin t + \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} x \frac{1}{\cos^2 \frac{t}{2} x 2}\right]$$

$$= a\left[-\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}}\right]$$

$$= a\left[-\sin t \frac{t}{\sin t}\right]$$

$$= a\left[\frac{-\sin^2 t + 1}{\sin t}\right]$$

$$= a\left[\frac{\cos^2 t}{\sin t}\right]$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= a \frac{\cos^2 t}{\sin t} \times \frac{1}{a.\cos t} = \cot t$$ [1]

Differentiating both sides w.r.t. x, we get

$$-\frac{1}{a} \text{cosec}^2 t. \sec t \text{ or } \frac{-1}{a \sin^2 t. \cos t}$$

$$\frac{d^2 y}{dx^2} = -\text{cosec}^2 t \frac{dt}{dx} \Rightarrow -\text{cosec}^2 t \frac{1}{a \cos t}$$ [1]

11. Let $A = (\sin x)^x$ and $B = \sin^{-1} \sqrt{x}$

Given $y = (\sin x)^x + \sin^{-1} \sqrt{x}$

$y = A + B$

$$\frac{dy}{dx} = \frac{dA}{dx} + \frac{dB}{dx} \quad ...(i)$$ [1]

As $A = (\sin x)^x$

Taking log on both sides we get

$\log A = \log(\sin x)^x$

$\log A = x \log(\sin x)$

Differentiating both sides w.r.t. x

$$\frac{1}{A}\times\frac{dA}{dx}=x\left(\frac{1}{\sin x}\times\cos x\right)+\log(\sin x)\times 1$$

$$\frac{dA}{dx}=A\left(x\cot x+\log(\sin x)\right) \quad [1]$$

$$=(\sin x)^{x}\left(x\cot x+\log(\sin x)\right) \quad ...(ii)$$

As $B=\sin^{-1}\sqrt{x}$

Differentiating both sides w.r.t. x

$$\frac{dB}{dx}=\frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\times\frac{d}{dx}\left(\sqrt{x}\right)$$

$$\frac{dB}{dx}=\frac{1}{\sqrt{1-x}}\times\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x-x^2}} \quad ....(iii) \quad [1]$$

From (i), (ii) and (iii) we get

$$\frac{dy}{dx}=(\sin x)^{x}\left(x\cot x+\log(\sin x)\right)+\frac{1}{2\sqrt{x-x^2}} \quad [1]$$

12. Given $x^m y^n=(x+y)^{m+n}$

Taking log on both sides we get

$$\log\left(x^m y^n\right)=\log(x+y)^{m+n} \quad [1]$$

$$m\log(\mathrm{x})+\mathrm{n}\log(\mathrm{y})=(m+n)\log(x+y)$$

Differentiating both sides w.r.t. 'x'

$$=m\times\frac{1}{x}+\mathrm{n}\times\frac{1}{y}\times\frac{dy}{dx}$$

$$=(m+n)\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$=\frac{m}{x}+\frac{n}{y}\frac{dy}{dx} \quad [1]$$

$$=\left(\frac{m+n}{x+y}\right)+\left(\frac{m+n}{x+y}\right)\frac{dy}{dx}$$

$$\Rightarrow\frac{m}{x}-\left(\frac{m+n}{x+y}\right)=\left(\frac{m+n}{x+y}\right)\frac{dy}{dx}-\frac{n}{y}\frac{dy}{dx}$$

$$\Rightarrow\left(\frac{m}{x}-\frac{m+n}{x+y}\right)=\left(\frac{m+n}{x+y}-\frac{n}{y}\right)\frac{dy}{dx}$$

$$\Rightarrow\left(\frac{m(x+y)-x(m+n)}{x(x+y)}\right)$$

$$=\left(\frac{y(m+n)-n(x+y)}{(x+y)\,y}\right)\frac{dy}{dx}$$

$$\Rightarrow\left(\frac{mx+my-mx-nx}{x(x+y)}\right)$$

$$=\left(\frac{my+ny-nx-ny}{(x+y)y}\right)\frac{dy}{dx}$$

$$\Rightarrow\left(\frac{my-nx}{x}\right)=\left(\frac{my-nx}{y}\right)\frac{dy}{dx}$$

$$\Rightarrow\frac{y}{x}=\frac{dy}{dx} \quad ...(i) \quad [1]$$

Differentiating again w.r.t. x we get

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{x\times\frac{dy}{dx}-y\times 1}{x^2}$$

From (i) we get

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{x\times\frac{y}{x}-y\times 1}{x^2}$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{y-y}{x^2}=0 \quad [1]$$

13. $x=a\sin 2t(1+\cos 2t)$

Differentiating both sides w.r.t. 't', we have

$$\frac{dx}{dt}=a\sin 2t(-2\sin 2t)+(1+\cos 2t)\times(2a\cos 2t)$$

$$=2a\sin 2t\left(-\sin^2 2t+\cos 2t+\cos^2 2t\right)$$

$$=2a(\cos 2t+\cos 4t) \quad [1]$$

$$\left(\because\cos^2 A-\sin^2 A=\cos 2A\right)$$

$$y=b\cos 2t(1-\cos 2t)$$

Differentiating both sides w.r.t. 't' we have

$$\frac{dy}{dt} = b\left[\cos 2t(2\sin 2t) + (1-\cos 2t)(-2\sin 2t)\right]$$

$$= 2b(\sin 2t \times \cos 2t - \sin 2t + \sin 2t \times \cos 2t)$$

$$= 2b(2\sin 2t \times \cos 2t - \sin 2t)$$

$$= 2b(\sin 4t - \sin 2t)$$

$$(\because 2\sin A\cos A = \sin 2A)$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{2b(\sin 4t - \sin 2t)}{2a(\cos 2t + \cos 4t)}$$ [1]

$$\left.\frac{dy}{dx}\right|_{\text{at } t=\frac{\pi}{4}} = \frac{b\left[\sin \pi - \sin\frac{\pi}{2}\right]}{a\left[\cos \pi + \cos\frac{\pi}{2}\right]}$$

$$= \frac{b(0-1)}{a(-1+0)}$$

$$= \frac{-b}{-a} = \frac{b}{a}$$ [1]

$$\left.\frac{dy}{dx}\right|_{\text{at } t=\frac{\pi}{3}} = \frac{b\left[\sin\frac{4\pi}{3} - \sin\frac{2\pi}{3}\right]}{a\left[\cos\frac{4\pi}{3} + \cos\frac{2\pi}{3}\right]}$$

$$= \frac{b\left(\frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)}{a\left(\frac{-1}{2} - \frac{1}{2}\right)}$$

$$= \frac{b\left(\frac{-2\sqrt{3}}{2}\right)}{a\left(\frac{-2}{2}\right)}$$

$$= \frac{b}{a}\sqrt{3}$$ [1]

14. $x = a\cos\theta + b\sin\theta$

Differentiating both sides w.r.t. 'θ', we have,

$$\frac{dx}{d\theta} = a(-\sin\theta) + b\cos\theta$$

$$= -(a\sin\theta - b\cos\theta) = -y$$ [1]

$$y = a\sin\theta - b\cos\theta$$

Differentiating both sides w.r.t. θ, we have,

$$\frac{dy}{d\theta} = a\cos\theta + b\sin\theta = x$$ [1]

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = x \times \frac{1}{-y} = -\frac{x}{y} \quad \text{...(i)}$$

$$\therefore y\frac{dy}{dx} = -x$$

$$\therefore y\frac{dy}{dx} + x = 0$$ [1]

Differentiating both sides w.r.t. 'x', we have,

$$y\frac{d^2y}{dx^2} + \frac{dy}{dx}.\frac{dy}{dx} + 1 = 0$$

$$y\frac{d^2y}{dx^2} + \left(-\frac{x}{y}\right).\frac{dy}{dx} + 1 = 0 \quad \text{....(from (i))}$$

Multiplying both sides by y, we have,

$$y^2\frac{d^2y}{dx^2} - \text{x}.\frac{dy}{dx} + y = 0$$ [1]

15. $y = x^x$

Taking log on both sides

$$\log y = \log x^x$$

$$\log y = x\log x$$ [1]

Differentiating both sides w.r.t. 'x', we get

$$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + \log x \times 1$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (1 + \log x) \quad \text{...(i)}$$

$$\Rightarrow \frac{dy}{dx} = y(1 + \log x)$$ [1]

Again Differentiating both sides w.r.t. 'x', we get

$$\frac{d^2y}{dx^2} = y \times \left(\frac{1}{x}\right) + (1 + \log x) \times \frac{dy}{dx}$$ [1]

$$\frac{d^2y}{dx^2} = \frac{y}{x} + \frac{1}{y} \times \frac{dy}{dx} \times \frac{dy}{dx} \qquad [\text{From (i)}]$$

$$\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$$ [1]

16. Let $y = \tan^{-1}\frac{\sqrt{1-x^2}}{x}$

Put $x = \sin\theta$

$$y = \tan^{-1}\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$$ [1]

$$y = \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$y = \tan^{-1}(\cot\theta)$$

$$y = \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \theta\right)\right)$$

$$y = \frac{\pi}{2} - \theta$$

Differentiating both sides w.r.t. 'θ', we have,

$$\frac{dy}{d\theta} = -1 \qquad \text{...(i)}$$

Now, let $z = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$ [1]

Put $x = \sin\theta$

$$z = \cos^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$$z = \cos^{-1}(2\sin\theta\cos\theta)$$

$$z = \cos^{-1}(\sin 2\theta)$$

$$z = \cos^{-1}\left(\cos\left(\frac{\pi}{2} - 2\theta\right)\right)$$

$$z = \frac{\pi}{2} - 2\theta$$ [1]

Differentiating both sides w.r.t. 'θ', we have,

$$\frac{dz}{d\theta} = -2 \qquad \text{...(i)}$$

Now, from equation (i) and (ii)

$$\frac{dy}{dz} = \frac{\frac{dy}{d\theta}}{\frac{dz}{d\theta}} = \frac{-1}{-2} = \frac{1}{2}$$ [1]

17. Given $y = \left(\tan^{-1} x^2\right)$

Differentiating both the sides w.r.t. 'x'.

$$\frac{dy}{dx} = 2\left(\tan^{-1} x\right) \times \frac{1}{1+x^2}$$ [1]

$$\left(1+x^2\right)\frac{dy}{dx} = 2\tan^{-1} x$$ [1]

Differentiating both the sides w.r.t. 'x'.

$$\left(1+x^2\right)\frac{d^2y}{dx^2} + \frac{dy}{dx}(2x) = \frac{2}{1+x^2}$$ [1]

$$\therefore \quad \left(x^2+1\right)^2\frac{d^2y}{dx^2} + 2x\left(x^2+1\right)\frac{dy}{dx} = 2$$ [1]

18. Given $\sin y = x\sin(a+y)$

$$\therefore \quad \frac{\sin y}{\sin(a+y)} = x$$ [1]

Differentiating both sides w.r.t. 'y', we get

$$\frac{\sin(a+y).\cos y - \sin y.\cos(a+y)}{\sin^2(a+y)} = \frac{dx}{dy}$$ [1]

$$\therefore \quad \frac{\sin((a+y)-y)}{\sin^2(a+y)} = \frac{dx}{dy}$$ [1]

$$\therefore \quad \frac{\sin a}{\sin^2(a+y)} = \frac{dx}{dy}$$ [1]

19. Given : Let $y = x^{x\cos x} + \left(\frac{x^2+1}{x^2-1}\right)$

Let $y = A + B$

$$\frac{dy}{dx} = \frac{dA}{dx} + \frac{dB}{dx}$$ [1]

$$A = x^{x\cos x}$$

Taking log on both sides, we get

$\log A = x\cos x\log x$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{A}\frac{dA}{dx} = x\cos x.\frac{1}{x} + x\log x(-\sin x) + \cos x\log x$$

$$\frac{dA}{dx} = A[\cos x - x\log x\sin x + \cos x\log x$$

$$\Rightarrow \frac{dA}{dx} = x^{x\cos x}[\cos x - x\log x\sin x + \cos x\log x]$$

[1]

$$B = \frac{X^2+1}{X^2-1}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dB}{dx} = \frac{\left(x^2 - 1.2x - \left(x^2+1\right).2x\right)}{\left(x^2-1\right)^2}$$

$$\frac{dB}{dx} = \frac{2x\left(x^2 - 1 - x^2 - 1\right)}{\left(x^2-1\right)^2}$$

$$\Rightarrow \frac{dB}{dx} = \frac{-4x}{\left(x^2-1\right)^2}$$ [1]

$$\frac{dy}{dx} = x^{x\cos x}(\cos\cos x - x\log x\sin x + \cos x\log x)$$

$$-\frac{4x}{\left(x^2-1\right)^2}$$ [1]

20. Given : $x = a(\theta - \sin\theta)$

Differentiating both sides w.r.t. 'θ' we get

$$\frac{dx}{d\theta} = a(1-\cos\theta)$$ [1]

$y = a(1+\cos\theta)$

Differentiating both sides w.r.t. 'θ' we get

$$\frac{dy}{d\theta} = a(-\sin\theta)$$ [1]

Now $\frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx}$

$$= -a\sin\theta\frac{1}{a(1-\cos\theta)}$$

$$= \frac{-\sin\theta}{1-\cos\theta} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$\therefore \quad \frac{dy}{dx} = -\cot\frac{\theta}{2}$$ [1]

Differentiating both sides w.r.t. 'x', we get

$$\frac{d^2y}{dx^2} = -\left(-\operatorname{cosec}^2\frac{\theta}{2}\right)\frac{1}{2}\frac{d\theta}{dx}$$

$$= \frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}\frac{1}{a(1-\cos\theta)}$$

$$= \frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}\frac{1}{a\left(2\sin^2\frac{\theta}{2}\right)}$$

$$= \frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2}$$ [1]

21. Given : $x = a(2\theta - \sin 2\theta)$ and $y = a(1-\cos 2\theta)$,

$x = a(2\theta - \sin 2\theta)$

Differentiating both sides w.r.t. 'θ' we get,

$$\frac{dx}{d\theta} = a\left(2\frac{d\theta}{d\theta} - \frac{d\sin 2\theta}{d\theta}\right)$$

$$\frac{dx}{d\theta} = a(2 - 2\cos 2\theta) \quad ...(i)$$ [1]

$y = a(1-\cos 2\theta)$,

Differentiating both sides w.r.t. 'θ' we get,

$$\frac{dy}{d\theta} = a\left(\frac{d}{d\theta}(1) - \frac{d}{d\theta}\cos 2\theta\right)$$

$$\frac{dy}{d\theta} = 2a\sin 2\theta \quad ...(ii)$$ [1]

We have,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$ [1]

From (1) and (2) we get,

$$\frac{dy}{dx}=\frac{2a\sin 2\theta}{a(2-2\cos 2\theta)}$$

$$\frac{dy}{dx}=\frac{2a\sin 2\theta}{2a(1-\cos 2\theta)}$$

Using $1-\cos 2\theta=2\sin^2\theta$, $\sin 2\theta=2\sin\theta\cos\theta$

$$\frac{dy}{dx}=\frac{4a\sin\theta\cos\theta}{4a\sin^2\theta}$$

$$\frac{dy}{dx}=\frac{\cos\theta}{\sin\theta}=\cot\theta$$

Now $\frac{dy}{dx}$ at $\theta=\frac{\pi}{3}$

$$\therefore\ \frac{dy}{dx}=\cot\left(\frac{\pi}{3}\right)=\frac{1}{\sqrt{3}}$$ [1]

22. Given : $\left(x^2+y^2\right)^2=xy$

Differentiating both sides by x we get

$$2\left(x^2+y^2\right)\left(2x+2y\frac{dy}{dx}\right)=x\frac{dy}{dx}+y\frac{d}{dx}(x)$$ [1]

$$4x^3+4xy^2+4x^2y\frac{dy}{dx}+4y^3\frac{dy}{dx}=x\frac{dy}{dx}+y$$

$$4x^2y\frac{dy}{dx}+4y^3\frac{dy}{dx}-x\frac{dy}{dx}=y-4x^3-4xy^2$$ [1]

$$\frac{dy}{dx}\left(4x^3y+4y^3-x\right)=y-4x^3-4xy^2$$ [1]

$$\frac{dy}{dx}=\frac{\left(y-4x^3-4xy^2\right)}{\left(4x^3y+4y^3-x\right)}$$ [1]

23. Given : $y=\sin(\sin x)$, ...(i)

Differentiating both sides w.r.t. x, we get,

$$\frac{dy}{dx}=\cos(\sin x)\times\cos x \quad ...(ii)$$ [1]

(Using Chain rule)

Differentiating both sides w.r.t. 'x', (above equation).

$$\frac{d^2y}{dx^2}=\cos x\times\frac{d}{dx}(\cos(\sin x))+\cos(\sin x)\frac{d}{dx}(\cos x)$$

[1]

$$\frac{d^2y}{dx^2}=\cos x(-\sin(\sin x)\cos x)+\cos(\sin x)(-\sin x)$$

$$\frac{d^2y}{dx^2}=-\cos^2 x\sin(\sin x)-\sin x\cos(\sin x)$$

From 1 we get

$$\frac{d^2y}{dx^2}=-\cos^2 x(y)-\sin x\cos(\sin x)$$ [1]

Multiply and divide the second part by cos x we get

$$\frac{d^2y}{dx^2}=-\cos^2 x(y)-\sin x\cos(\sin x)\frac{\cos x}{\cos x}$$

$$\frac{d^2y}{dx^2}=-\cos^2 x(y)-\tan x\cos(\sin x)\cos x$$

From (2) we get

$$\frac{d^2y}{dx^2}=-\cos^2 x(y)-\tan x\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2}-\tan x\frac{dy}{dx}-y\cos^2 x=0$$ [1]

Proved.

24. $y=\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

Applying the identity $1+\cos x=2\cos^2\frac{x}{2}$ and

$$\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}$$ [1]

$$y=\tan^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$$=\tan^{-1}\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right)$$ [1]

$$= \tan^{-1}\left(\cot\frac{x}{2}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right) \quad [1]$$

$$y = \frac{\pi}{2}-\frac{x}{2} \quad [1]$$

Differentiating both sides with respect to x we get,

$$\frac{dy}{dx} = -\frac{1}{2} \quad [1]$$

25. Let $y = (x\cos x)^x + (x\sin x)^{\frac{1}{x}}$

$y = u + v$

Differentiating w.r.t. x both sides we get

$$\frac{dy}{dx} = \frac{du}{dx}+\frac{dv}{dx} \quad [1]$$

$$u = (x\cos x)^x$$

Taking log both sides

$$\log u = \log(x\cos x)^x$$

On differentiation

$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\left[\log(x\cos x)\right] + \log(x\cos x)\frac{d}{dx}(x)$$

$$\frac{1}{u}\frac{du}{dx} = x\left(\frac{1}{x\cos x}\right)\frac{d}{dx}\left[x\cos x\right] + \log(x\cos x).1$$

$$\frac{1}{u}\frac{du}{dx} = x\left(\frac{1}{x\cos x}\right)\left[x\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(x)\right] + \log(x\cos x)$$

$$\frac{1}{u}\frac{du}{dx} = x\left(\frac{1}{x\cos x}\right)\left[x(-\sin x) + \cos x.1\right] + \log(x\cos x)$$

$$\frac{1}{u}\frac{du}{dx} = \left(\frac{1}{\cos x}\right)\left[-x\sin x + \cos x\right] + \log(x\cos x)$$

$$\frac{1}{u}\frac{du}{dx} = \left[-x\tan x + 1\right] + \log(x\cos x)$$

$$\frac{1}{u}\frac{du}{dx} = \left[1 - x\tan x\right] + \log(x\cos x)$$

$$\frac{du}{dx} = u.\left[(1 - x\tan x) + \log(x\cos x)\right] \quad [1]$$

$$\frac{du}{dx} = (x\cos x)^x.\left[(1 - x\tan x) + \log(x\cos x)\right] \quad ...(i)$$

Now,

$$v = (x\sin x)^{\frac{1}{x}}$$

Taking log both sides

$$\log v = \frac{1}{x}\log(x\sin x)$$

$$\log v = \frac{1}{x}(\log x + \log\sin x)$$

$$\log v = \frac{1}{x}(\log x) + \frac{1}{x}(\log\sin x)$$

On differentiation

$$\frac{1}{v}\frac{dv}{dx} = \frac{1}{x}\frac{d}{dx}\left[\log x\right] + \log(x)\frac{d}{dx}\left(\frac{1}{x}\right) + \log(\sin x).\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}\frac{d}{dx}(\log\sin x)$$

$$\frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2} + \log(x)\left(-\frac{1}{x^2}\right) + \log(\sin x).\left(-\frac{1}{x^2}\right) + \frac{1}{x}\left(\frac{1}{\sin x}\right)\frac{d}{dx}(\sin x)$$

$$\frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2} - \frac{\log(x)}{x^2} - \frac{\log(\sin x)}{x^2} + \frac{1}{x}\left(\frac{1}{\sin x}\right)\cos x$$

$$\frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2} - \frac{\log(x)}{x^2} - \frac{\log(\sin x)}{x^2} + \frac{\cot x}{x}$$

$$\frac{1}{v}\frac{dv}{dx} = \frac{1 - \log(x) - \log(\sin x) + x\cot x}{x^2}$$

$$\frac{dv}{dx} = v.\left[\frac{1 - \log(x\sin x) + x\cot x}{x^2}\right]$$

$$\frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}.\left[\frac{1 - \log(x\sin x) + x\cot x}{x^2}\right] \quad ...(ii) \quad [1]$$

Thus,

$$\frac{dy}{dx} = \frac{du}{dx}+\frac{dv}{dx}$$

From (i) and (ii) we get

$$\frac{dy}{dx} = (x\cos x)^x.\left[(1 - x\tan x) + \log(x\cos x)\right] + (x\sin x)^{\frac{1}{x}}.\left[\frac{1 - \log(x\sin x) + x\cot x}{x^2}\right] \quad [1]$$

26. $y = 3\cos(\log x) + 4\sin(\log x)$

$$\frac{dy}{dx} = -3\sin(\log x).\frac{1}{x} + 4\cos(\log x)\frac{1}{x}$$

$$\frac{dy}{dx} = \frac{-3\sin(\log x) + 4\cos(\log x)}{x}$$

$$y_1 = \frac{-3\sin(\log x) + 4\cos(\log x)}{x} \quad [1]$$

Now, $xy_1 = -3\sin(\log x) + 4\cos(\log x)$ [1]

On further differentiation

$$xy_2 + y_1 = -3\cos(\log x).\left(\frac{1}{x}\right) - 4\sin(\log x).\left(\frac{1}{x}\right) \quad [1]$$

$$xy_2 + y_1 = -\left(\frac{3\cos(\log x) + 4\sin(\log x)}{x}\right)$$

$$xy_2 + y_1 = -\frac{y}{x}$$

$$x^2 y_2 + xy_1 + y = 0 \quad [1]$$

Hence Proved.

27. $(\cos x)^y = (\cos y)^x$

Taking logarithm of both sides, we get

$$\log(\cos x)^y = \log(\cos y)^x$$

$$\therefore\ y\log(\cos x) = x\log(\cos y) \quad [1]$$

Differentiating both sides we get

$$\therefore\ y\frac{1}{\cos x}(-\sin x) + \log(\cos x)\ \frac{dy}{dx}$$

$$= x\ \frac{1}{cosy}\ (-\sin y)\ \frac{dy}{dx} + \log(\cos y)$$

$$\therefore\ -\frac{y\sin x}{\cos x} + \log\cos x.\frac{dy}{dx}$$

$$= -\left(\frac{x\sin y}{cosy}\right).\frac{dy}{dx} + \log\cos y \quad [1]$$

$$\therefore\ -\frac{y\sin x}{\cos x} + \log\cos x.\frac{dy}{dx}$$

$$= -\left(\frac{x\sin y}{cosy}\right).\frac{dy}{dx} + \log\cos y$$

$$\Rightarrow \frac{dy}{dx}\left[\log(\cos x) + \frac{x\sin y}{cosy}\right] = \log(\cos y) + y\frac{\sin x}{\cos x} \quad [1]$$

$$\Rightarrow \frac{dy}{dx}\left[\log(\cos x) + \frac{x\sin y}{cosy}\right] = \log(\cos y) + y\frac{\sin x}{\cos x} \quad [1]$$

28. $y^x = e^{y-x}$,

$\Rightarrow x\log_e y = y - x$ (i) [1]

Differentiating w.r.t. x

$$\therefore\ \log_e y + x.\frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx} - 1$$

$$\therefore\ \log_e y + 1 = \frac{dy}{dx}\left(1 - \frac{x}{y}\right)\left\{\text{From (1)}\ \frac{x}{y} = \frac{1}{1 + lor_e y}\right\}$$

[1]

$$\therefore\ \log_e y + 1 = \frac{dy}{dx}\left(1 - \frac{1}{1 + \log_e y}\right) \quad [1]$$

$$\therefore\ (\log_e y + 1) = \frac{dy}{dx}\left(\frac{\log_e y}{1 + \log_e y}\right)$$

$$\therefore\ \frac{dy}{dx} = \frac{(1 + \log_e y)^2}{\log_e y} \quad [1]$$

29. $y = \sin^{-1}\left[\dfrac{2^{x+1}.3^x}{1 + (36)^2}\right]$

$$y = \sin^{-1}\left[\frac{2^x.2.3^x}{1 + (36)^2}\right] \quad [1]$$

$$y = \sin^{-1}\left[\frac{2.6^x}{1 + (6)^{2x}}\right]$$

$$y = 2\tan^{-1}(6)^x \quad [1]$$

$$\frac{dy}{dx} = \frac{2}{1 + (6)^{2x}}.6^x\log 6 \quad [1]$$

$$\frac{dy}{dx} = \frac{2.6^x\log 6}{1 + (36)^2} \quad [1]$$

30. $x = a\cos^3\theta$ and $y = a\sin^3\theta$

Differentiating w.r.t. θ we get.

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \text{ and } \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta \quad [1]$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\therefore \frac{dy}{dx} = -\tan\theta \quad [1]$$

$$\therefore \frac{d^2y}{dx^2} = -\sec^2\theta\frac{d\theta}{dx}$$

$$\therefore \frac{d^2y}{dx^2} = -\sec^2\theta\frac{1}{(-3a\cos^2\theta.\sin\theta)} \quad [1]$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{3a}\sec^4\theta.cosec\theta$$

$$\left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{6}} = \frac{1}{3a}\left(\frac{2}{\sqrt{3}}\right)^4.2$$

$$= \frac{32}{27a} \quad [1]$$

31. $y = ae^\theta(\sin\theta + \cos\theta)$

$x = ae^\theta(\sin\theta - \cos\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \quad [1]$$

Now,

$$\frac{dy}{d\theta} = ae^\theta(\cos\theta - \sin\theta) + ae^\theta(\sin\theta + \cos\theta) \quad [1]$$

$$= 2ae^\theta(\cos\theta)(\text{Applying product rule}) \quad [1]$$

$$\frac{dx}{d\theta} = ae^\theta(\cos\theta + \sin\theta) + ae^\theta(\sin\theta - \cos\theta)$$

$$= 2ae^\theta(\sin\theta)$$

Substituting the values of

$\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ in (1).

$$\frac{dy}{dx} = \frac{2ae^\theta\cos\theta}{2ae^\theta\sin\theta} = \cos\theta$$

Now $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$

$$[\cot\theta]_{\theta=\frac{\pi}{4}} = \cot\frac{\pi}{4} = 1 \quad [1]$$

32. Let $y = \cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right)$

$$= \cos^{-1}\left(\frac{x^2 - 1}{x^2 + 1}\right) \quad [1]$$

$$= \pi - \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right) \quad [1]$$

$$= \pi - 2\tan^{-1}x \quad [1]$$

$$\frac{dy}{dx} = -\frac{2}{1 + x^2} \quad [1]$$

33. Let $y = \cos^{-1}\left\{\sin\sqrt{\frac{1+x}{2}}\right\} + x^x$

Let $u = \cos^{-1}\left\{\sin\sqrt{\frac{1+x}{2}}\right\}; v = x^x$ [1]

$\therefore\ y = u + v$

Differentiating w.r.t. 'x' we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$u = \cos^{-1}\left\{\sin\sqrt{\frac{1+x}{2}}\right\}$$

$$= \cos^{-1}\left[cos.\left(\frac{\pi}{2} - \sqrt{\frac{1+x}{2}}\right)\right]$$

$$= \frac{\pi}{2} - \sqrt{\frac{1+x}{2}}$$

$$\frac{du}{dx} = -\frac{1}{2\sqrt{2}\sqrt{1+x}} \ldots\ldots(1)$$ [1]

$$v = x^x$$

$$\log v = x \log x$$

$$\frac{1}{v}\frac{dv}{dx} = x.\frac{1}{x} + 1 logx = 1 + logx$$

$$\frac{dv}{dx} = x^x(1 + \log x)\ldots\ldots(2)$$ [1]

$$\frac{dy}{dx} = -\frac{1}{2\sqrt{2}\sqrt{1+x}} + x^x\left(1 + logx\right)$$

$$\left(\frac{dy}{dx}\right)_{at\, x=1} = -\frac{1}{4} + 1$$

$$= \frac{3}{4}$$ [1]

34. $$\frac{dx}{dt} = \alpha\left[-2sin2t\ sin2t + 2cos2t\left(1 + cos2t\right)\right]$$ [1]

$$\frac{dy}{dt} = \beta\left[2sin2t\ cos2t - \left(1 - cos2t\right).2sin2t\right]$$ [1]

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{\beta\left[2sin2t\ cos2t - \left(1 - cos2t\right).2sin2t\right]}{\alpha\left[-2sin2t\ sin2t + 2cos2t\left(1 + cos2t\right)\right]}$$ [1]

$$= \frac{\beta(2\sin 4t - 2\sin 2t)}{\alpha(2\cos 4t\ 2\cos 2t)}$$

$$= \frac{\beta}{\alpha}.\frac{2\cos 3t \sin t}{2\cos 3t \cos t}$$

$$= \frac{\beta}{\alpha}\tan t$$ [1]

Smart Notes ....

Smart Notes ....

CHAPTER 6

Application of Derivatives

Chapter Analysis of Previous Three Years' Board Exams

Number of Questions Asked in Exams

Topics	2018		2017		2016	
	Delhi	**All India**	**Delhi**	**All India**	**Delhi**	**All India**
Maxima and Minima	4 marks	4 marks	2, 6 marks	2, 6 marks	6 marks	6 marks
Increasing, Decreasing Function			2 marks	2 marks		
Equation of Normal and Tangent	4 marks	4 marks			4 marks	4 marks
Rolles Theorem				2 marks		
Marginal Cost	2 marks	2 marks				

[TOPIC 1] Rate of Change, Increasing and Decreasing Functions and Approximations

Summary

- **Rate of Change of Bodies** representing $\frac{dy}{dx}$ as a rate measure:

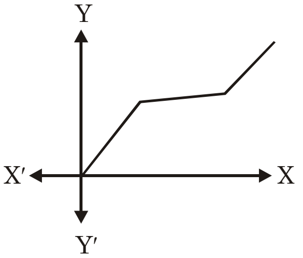
  Let us take two variables x and y that vary with respect to another variable says, i.e. if $x = f(s)$ and $y = g(s)$, then by applying the chain rule, we have

  $$\frac{dy}{dx} = \frac{\frac{dy}{ds}}{\frac{dx}{ds}}, \text{ if } \frac{dx}{ds} \neq 0.$$

  Thus, the rate of change of y with respect to x can be calculated using the rate of change of y and that of x both with respect to s.

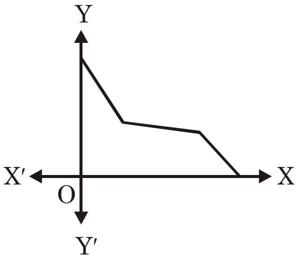
- **Increasing and decreasing functions**
  - ➢ A function is said to be increasing when the y value increases as the x value increases.

    Example:



  - ➢ A function is said to be increasing when the y value decreases as the x value increases.

    Example:

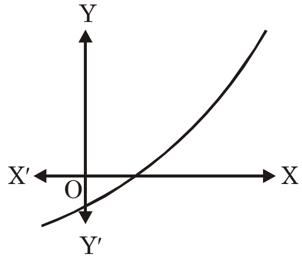


  - ➢ f is strictly increasing if

    $x_1 < x_2$,

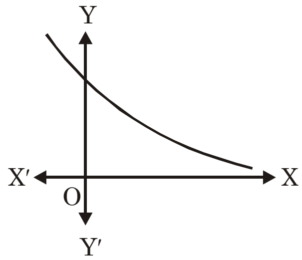
    $\Rightarrow \quad f(x_1) < f(x_2)$.

    Example:



  - ➢ f is strictly decreasing if $x_1 < x_2, \Rightarrow f(x_1) < f(x_2)$.

    Example:



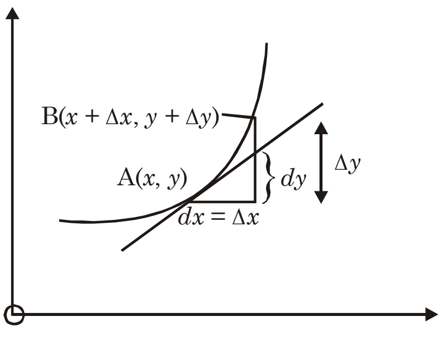
  - ➢ Let f be continuous on [a, b] and differentiable on the open interval (a, b). Then:

    f is increasing in [a, b] if $f'(x) > 0$ for each $x \in (a, b)$

    f is decreasing in [a, b] if $f'(x) < 0$ for each $x \in (a, b)$

    f is a constant function in [a, b] if $f'(x) = 0$ for each $x \in (a, b)$

- **Approximations**
  - ➢ Let the given function be $y = f(x)$. Δx denotes a small increment in x.
  - ➢ The corresponding increment in y is given by $\Delta y = f(x + \Delta x) - f(x)$
  - ➢ Differential of y, denoted by dy is $dy = \left(\frac{dy}{dx}\right)\Delta x$



PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

2 Marks Questions

1. Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on

[ALL INDIA 2012]

2. The volume of a sphere is increasing at the rate of 3 cubic centimeters per second. Find the rate of increase of its surface area, when the radius is 2 cm.

[DELHI 2017]

3. Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing on R.

[DELHI 2017]

4 Marks Questions

4. Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is

(a) Strictly increasing

(b) Strictly decreasing

[DELHI 2014]

5. Sand is pouring from a pipe at the rate of 12 cm^2/s. The falling and sand forms a cone on the ground in such a way the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm ?

[DELHI 2011]

6. Find the intervals in which the function

$$f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12 \text{ is}$$

(a) Strictly increasing

(b) Strictly decreasing

[DELHI 2018]

7. Prove that the function f given by $f(x) = \log \sin x$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

[ALL INDIA 2011]

8. Using differentials, find the approximate value of $\sqrt{49.5}$.

[DELHI 2012]

9. Find the values of x for which $y = [x(x-2)]^2$ is an increasing function.

[ALL INDIA 2014]

10. Prove that $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function of θ on $\left[0, \frac{\pi}{2}\right]$.

[ALL INDIA 2016]

6 Marks Question

11. Find the intervals in which $f(x) = \sin 3x - \cos 3x$, $0 < x < \pi$ is strictly increasing or strictly decreasing.

[DELHI 2016]

Solutions

1. $f(x) = x^3 - 3x^2 + 6x - 100$

$f'(x) = 3x^2 - 6x + 6$

$f'(x) = 3(x^2 - 2x + 2)$

$f'(x) = 3\left[(x-1)^2 + 1\right]$

$f'(x) > 0 \ \forall x \in R$

So, is increasing on R. [2]

2. Let r be the radius, v be the volume, and s be the surface area of the sphere.

Then volume $v = \frac{4}{3}\pi r^3$

Surface area $s = 4\pi r^2$

Given $\frac{dv}{dt} = 3\,\text{cm}^3/\text{s}$ ...(i)

$v = \frac{4}{3}\pi r^3$

Differentiating both sides w.r.t "t", we have

$$\frac{dv}{dt} = \frac{4}{3}\pi \times 3r^2 \frac{dr}{dt}$$

$\Rightarrow \frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$

From (1) we get

$3 = 4\pi r^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{3}{4\pi r^2}$ [1]

Now surface area $s = 4\pi r^2$

Differentiating both sides w.r.t "t", we have

$\frac{ds}{dt} = 4\pi \times 2r \frac{dr}{dt}$

From (1) we get,

$\frac{ds}{dt} = 4\pi \times 2r\left(\frac{3}{4\pi r^2}\right)$

$\frac{ds}{dt} = 8\pi r\left(\frac{3}{4\pi r^2}\right)$

$= \frac{6}{r}$

Substituting $r = 2\,cm$

$\frac{ds}{dt} = \frac{6}{2} = 3\,cm^2/\text{sec}$

Surface area is changing at rate of 3cm²/sec. [1]

3. Given $f(x) = 4x^3 - 18x^2 + 27x - 7$

Differentiating both sides w.r.t x we get,

$f'(x) = 12x^2 - 36x + 27$

$f'(x) = 3\left(4x^2 - 12x + 9\right)$

$f'(x) = 3\left((2x)^2 - 2(3)(2x) + 3^2\right)$

$f'(x) = 3(2x - 3)^2$

Given that $f(x)$ is always increasing on R.

$\Rightarrow 3(2x-3)^2 \geq 0 \ \forall \ x \in R$

$\Rightarrow f'(x) \geq 0 \ \forall \ x \in R$

Thus, $f(x)$ is increasing on R. [2]

4. We have, $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

Differentiating both sides w.r.t. x, we get,

$f'(x) = 12x^3 - 12x^2 - 24x$ [1]

$f'(x) = 12x(x^2 - x - 2)$

$f'(x) = 12x(x^2 - 2x + x - 2)$

$f'(x) = 12x(x(x-2) + 1(x-2))$

$f'(x) = 12x(x+1)(x-2))$

When $f'(x) = 0 \therefore x = -1, 0, 2$ [1]

Intervals	Checking points	Values of 12	x	x + 1	x − 1	Sign of $f'(x)$	Nature of $f(x)$
$(-\infty, -1)$	-2	+	–	–	–	< 0	Strictly decreasing
$(-1, 0)$	-0.5	+	–	+	–	> 0	Strictly increasing
(0, 2)	1	+	+	+	–	< 0	Strictly decreasing
$(2, \infty)$	3	+	+	+	+	> 0	Strictly increasing

[1]

$f(x)$ is strictly increasing in $(-1, 0)$ and $(2, \infty)$.

$f(x)$ is strictly decreasing $(-\infty, -1)$ in $(0, 2)$ and. [1]

5. Given: Let x be the radius, h be the height and v be the volume of the cone.

To find : $\dfrac{dh}{dt}$

$\dfrac{dv}{dt} = 12\text{cm}^3/\text{s}, \quad h = 4\text{cm}$

$h = \dfrac{1}{6}r$

$\therefore\ 6h = r$ [1]

Volume of a cone $v = \dfrac{1}{3}\pi r^2 h$,

$= \dfrac{\pi}{3}\left(36h^2\right)h$ [1]

$v = 12\pi h^3$

Differentiating both sides w.r.t. 't', we get

$\dfrac{dv}{dt} = 36\pi h^2 \dfrac{dh}{dt}$ [1]

$12 = 36\pi h^2 = \dfrac{dh}{dt}$

$\dfrac{dh}{dt} = \dfrac{1}{3\pi h^2} = \dfrac{1}{3\pi(16)}$ at h = 4cm

$\dfrac{dh}{dt} = \dfrac{1}{48\pi}\text{cm/sec}$

Increase in height = $\dfrac{1}{48\pi}\text{cm/sec}$. [1]

6. Given $f(x) = \dfrac{x^4}{4} - x^3 - 5x^2 + 24x + 12$

Differentiating both sides w.r.t. x, we get

$f'(x) = x^3 - 3x^2 - 10x + 24$

On checking the values we get $x = 2$ is a zero of equation.

As $f'(2) = (2)^3 - 3(2)^2 - 10(2) + 24 = 32 - 32 = 0$ [1]

Thus, $(x-2)$ is a factor of $f'(x)$.

Using long division method we get,

$$\begin{array}{r|l} & x^2 - x - 12 \\ \hline x-2 & x^3 - 3x^2 - 10x + 24 \\ & x^3 - 2x^2 \\ & - \quad + \\ \hline & -x^2 - 10x + 24 \\ & -x^2 + 2x \\ & + \quad - \\ \hline & -12x + 24 \\ & -12x + 24 \\ & + \quad - \\ \hline & 0 \end{array}$$

$f'(x) = (x-2)\left(x^2 - x - 12\right)$

$f'(x) = (x-2)\left(x^2 - 4x + 3x - 12\right)$

$f'(x) = (x-2)\left(x(x-4) + 3(x-4)\right)$

$f'(x) = (x-2)(x+3)(x-4)$

$f'(x) = 0$ for $x = -3, 2, 4$ [1]

Intervals	Checking point	All values			Sign of function	Nature of function
		$(x-2)$	$(x+3)$	$(x-4)$		
$(-\infty, -3)$	-5	–	–	–	< 0	Strictly decreasing
$(-3, 2)$	0	–	+	–	> 0	Strictly Increasing
$(2, 4)$	3	+	+	–	< 0	Strictly Decreasing
$(4, \infty)$	6	+	+	+	> 0	Strictly Increasing

[1]

Thus, f is strictly increasing in the interval $(-3,2)\cup(4,\infty)$ and strictly decreasing in the interval $(-\infty,-3)\cup(2,4)$. [1]

7. We have,

$f(x) = \log \sin x$

$f'(x) = \frac{1}{\sin x} . \cos x = \cot x$ [2]

Interval	– or +	Increase/ Decrease
$\left(0, \frac{\pi}{2}\right)$	$f'(x) = \cot x > 0$ Positive	Increase
$\left(\frac{\pi}{2}, \pi\right)$	$f'(x) = \cot x < 0$ Negative	Decrease

[2]

8. Let $f(x) = \sqrt{x}$, where $x = 49$ let $\Delta x = 0.5$

$\therefore\ f(x + \Delta x) = \sqrt{(x + \Delta x)} = \sqrt{49.5}$ [1]

Now by definition, approximately we can write

$f'(x) = \sqrt{x} = \sqrt{49} = 7$ [1]

$\delta x = 0.5$

$f'(x) = \frac{1}{2\sqrt{x}}$

$= \frac{1}{2\sqrt{49}}$

$= \frac{1}{14}$ [1]

Putting these values in (i) , we get.

$\frac{1}{14} = \frac{\sqrt{49.5} - 7}{0.5}$

$\therefore\ \sqrt{49.5} = \frac{0.5}{14} + 7$

$= \frac{0.5 + 98}{14}$

$= \frac{98.5}{14} = \mathbf{7.036}$ [1]

9. Let $f(x) = [x(x-2)]^2$

We know, for increasing function we have $f'(x) \geq 0$ [1]

$\therefore f'(x) = 2[x(x-2)]\left[\frac{d}{dx} x(x-2)\right]$

Or $f'(x) = 2[x(x-2)\frac{d}{dx}(x^2 - 2x)$ [1]

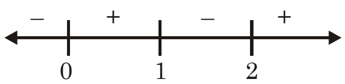
$= 2x(x-2)(2x-2)$

$= 4x(x-2)(x-1)$

For $f'(x) \geq 0$

i.e., $4x(x-1)\ (x-2) \geq 0$ [1]

the values of x are



$X \in [0,1] \cup [2, \infty)$ [1]

10. $y = \frac{4\sin\theta}{2 + \cos\theta} - \theta,\ \ \theta \in \left[0, \frac{\pi}{2}\right]$

Differentiating w.r.t θ, we get

$\frac{dy}{d\theta} = 4\left\{\frac{(2 + \cos\theta)\cos\theta - \sin\theta(-\sin\theta)}{(2 + \cos\theta)^2}\right\} - 1$ [1]

$= 4\left\{\frac{2\cos\theta + 1}{(2 + \cos\theta)^2}\right\} - 1$

$= \frac{8\cos\theta + 4 - (2 + \cos\theta)^2}{(2 + \cos\theta)^2}$

$= \frac{\cos\theta(4 - \cos\theta)}{(2 + \cos\theta)^2}\ \left[\because 0 \leq \cos\theta \leq 1, \forall\theta \in \left[0, \frac{\pi}{2}\right]\right]$ [1]

$\Rightarrow \frac{\cos\theta(4 - \cos\theta)}{(2 + \cos\theta)^2} \geq 0; \forall\theta \in \left[0, \frac{\pi}{2}\right]$ [1]

$\Rightarrow \frac{dy}{d\theta} \geq 0; \forall\theta \in \left[0, \frac{\pi}{2}\right]$

Thus, y is increasing in

$\left[0, \frac{\pi}{2}\right]$ [1]

11. $f(x) = \sin 3x - \cos 3x$

Differentiating both sides w.r.t. x, we have

$f'(x) = 3\cos 3x + 3\sin 3x = 3(\cos 3x + \sin 3x)$ [1]

For strictly increasing or decreasing,

$f'(x) = 0$

$3\cos 3x + 3\sin 3x = 0$

$3\sin 3x = -3\cos 3x$ [1]

$\tan 3x = -1$

$3x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}$

$x = \frac{3\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$ [1]

Intervals	Checking points	Values of 3	Values of $(\cos 3x + \sin 3x)$	Sign of $f'(x)$	Nature of $f(x)$
$0 < x < \frac{\pi}{4}$	$\frac{\pi}{6}$	+	+	>0	Strictly increasing
$\frac{\pi}{4} < x < \frac{7\pi}{12}$	$\frac{\pi}{2}$	+	–	<0	Strictly decreasing
$\frac{7\pi}{12} < x < \frac{11\pi}{12}$	$\frac{5\pi}{6}$	+	+	>0	Strictly increasing
$\frac{11\pi}{12} < x < \pi$	$\frac{17\pi}{8}$	+	–	<0	Strictly decreasing

[1]

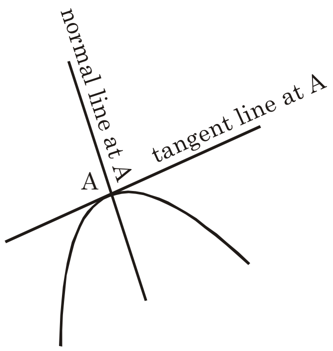
$f(x)$ is strictly increasing in $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$ [1]

$f(x)$ is strictly decreasing in $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right) \cup \left(\frac{11\pi}{12}, \pi\right)$ [1]

[TOPIC 2] Tangents and Normals

Summary

- A tangent line is defined as a straight line that touches the given function at only one point and it represents the instantaneous rate of change of function at the point.
- A normal line to a point (x, y) on a curve is the line that goes through the point (x, y) and is perpendicular to the tangent line.



- Slope or gradient of a line: If a line makes an angle θ with the positive direction of X axis in anti-clockwise direction, then tan θ is called the slope or gradient of the line.
- If a tangent line to the curve y = f(x) makes an angle θ with x-axis in the positive direction, then

  slope of the tangent = $\tan\theta = \frac{dy}{dx}$
- If slope of the tangent line is zero, then tan θ = 0 and so θ = 0 which means the tangent line is parallel to the x-axis. In this case, the equation of the tangent at the point is given by $(y = y_0)$
- If $\theta \to \frac{\pi}{2}$ then $\tan\theta \to \infty$, which means the tangent line is perpendicular to the x-axis, i.e., parallel to the y-axis. In this case, the equation of the tangent at (x_0, y_0) is given by $(x = x_0)$
- Equation of tangent at (x_1, y_1) is given by $(y - y_1) = m_T(x - x_1)$, where m_T is the slope of the tangent

  such that $m_N = \left[\frac{dy}{dx}\right]_{(x_1, y_1)}$
- Equation of normal at (x_1, y_1) is given by $(y - y_1) = m_N(x - x_1)$, where m_N is the slope of the normal

  such that $m_N = \frac{-1}{\left[\frac{dy}{dx}\right]_{(x_1, y_1)}}$
- Tangent and normal are perpendicular to each other, which gives us $m_T \times m_N = -1$
- If the slope of two different curves are m_1 and m_2, then the acute angle between them is given by

  $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1.m_2}\right|$
- The slope intercept form of the line is $y = mx + c$, where m is the slope of the given line.

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 2

4 Marks Questions

1. Show that the equation of normal at any point on the curve

   $x = 3\cos t - \cos^3 t$ and $y = 3\sin t - \sin^3 t$ is $4(y\cos^3 t - x\sin^3 t) = 3\sin 4t$

   [DELHI 2016]

2. Find the equations of the tangent and normal to the curve $x = a\sin^3\theta$ and $y = a\cos^3\theta$ at $\theta = \frac{\pi}{4}$.

   [DELHI 2014]

3. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to x-axis.

   [DELHI 2011]

4. Find the equations of the tangent and the normal, to the curve $16x^2 + 9y^2 = 145$ at the point (x_1, y_1), where $x_1 = 2$ and $y_1 > 0$.

   [DELHI 2018]

5. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.

[ALL INDIA 2011]

6. Find the point on the curve $y = x^3 - 11x + 5$ at which the equation of tangent is $y = x - 11$.

[ALL INDIA 2012]

7. Find the eqaution of the tangent and normal to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $\left(\sqrt{2}a, b\right)$.

[ALL INDIA 2014]

8. Find the equation of tangents to the curve $y = x^3 + 2x - 4$, which are perpendicular to line

[ALL INDIA 2016]

6 Marks Question

9. Find the equations of tangents to the curve $3x^2 - y^2 = 8$, which pass through the point $\left(\frac{4}{3}, 0\right)$.

[ALL INDIA 2013]

Solutions

1. $x = 3\cos t - \cos^3 t$

Differentiating both sides w.r.t, we have

$$\frac{dx}{dt} = -3\sin t - 3\cos^2 t(-\sin t)$$

$$= -3\sin t(1 - \cos^2 t) = -3\sin^3 t$$

$$y = 3\sin t - \sin^3 t$$

Differentiating both sides w.r.t t, we have

$$\frac{dy}{dt} = 3\cos t - 3\sin^2 t.\cos t$$

$$= 3\cos t(1 - \sin^2 t) \quad [1]$$

$$= 3\cos t(\cos^2 t) = 3\cos^3 t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{3\cos^3 t}{-3\sin^3 t} = \frac{-\cos^3 t}{\sin^3 t}$$

Slope of normal $\frac{-dx}{dy} = \frac{\sin^3 t}{\cos^3 t}$

Equation of normal is

$(y - y_1) = \text{Slope of normal}(x - x_1)$

$$y - (3\sin t - \sin^3 t) = \frac{\sin^3 t}{\cos^3 t}.\left[x - (3\cos t - \cos^3 t)\right]$$

$$\Rightarrow y\cos^3 t - 3\sin t\cos^3 t + \sin^3 t\cos^3 t$$

$$= x\sin^3 t - 3\sin^3 t\cos t + \sin^3 t\cos^3 t$$

$$\Rightarrow y\cos^3 t - x\sin^3 t = 3\sin t\cos^3 t - 3\sin^3 t\cos t \quad [½]$$

$$\Rightarrow y\cos^3 t - x\sin^3 t = 3\sin t\cos^3 t - 3\sin^3 t\cos t$$

$$\Rightarrow y\cos^3 t - x\sin^3 t = 3\sin t\cos t(\cos^2 t - \sin^2 t) \quad [½]$$

Multiplying both sides by

$$4(y\cos^3 t - x\sin^3 t) = 3 \times 2 \times 2\sin t\cos t(\cos 2t) \quad [½]$$

Using 2 sin A cos A = sin²A, cos²A – sin²A = cos2A [1]

$$4(y\cos^3 t - x\sin^3 t) = 3 \times 2\sin 2t\cos 2t$$

$$4(y\cos^3 t - x\sin^3 t) = 3\sin 4\text{t} \quad [½]$$

Hence, proved

2. when $\theta = \frac{\pi}{4}$

$$x = a\sin^3\theta = a\sin^3\frac{\pi}{4}$$

$$= a\left(\frac{1}{\sqrt{2}}\right)^3 = \frac{a}{2\sqrt{2}}$$

$$y = a\cos^3\theta = a\cos^3\frac{\pi}{4}$$

$$= a\left(\frac{1}{\sqrt{2}}\right)^3 = \frac{a}{2\sqrt{2}}$$

So, coordinates of the point of contact are

$$\left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right) \quad [1]$$

$x = a\sin^3\theta$ and $y = a\cos^3\theta$

Differentiating both sides w.r.t.θ, we get

$$\frac{dx}{d\theta} = 3a\sin^2\theta.\cos\theta \text{ and } \frac{dy}{d\theta} = 3a\cos^2\theta.(-\sin\theta)$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-3a\cos^2\theta.\sin\theta}{3a\sin^2\theta.\cos\theta} = -\cot\theta \quad [1]$$

Slope of tangent, $\left.\frac{dy}{dx}\right]_{\text{at}\,\theta=\frac{\pi}{4}} = -\cot\left(\frac{\pi}{4}\right) = -1$

Slope of normal = 1

Equation of tangent at $\left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right)$ is

$$y - \frac{a}{2\sqrt{2}} = -1\left(x - \frac{a}{2\sqrt{2}}\right)$$

$$y - \frac{a}{2\sqrt{2}} = -x + \frac{a}{2\sqrt{2}}$$

$$y + x = \frac{2a}{2\sqrt{2}} = \frac{a}{\sqrt{2}}$$

$\sqrt{2}(y+x) = a\ \sqrt{2}(y+x) = a$ [1]

Equation of normal at $\left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right)$ is

$$y - \frac{a}{2\sqrt{2}} = 1\left(x - \frac{a}{2\sqrt{2}}\right)$$

$$y - \frac{a}{2\sqrt{2}} = x - \frac{a}{2\sqrt{2}}$$

$$y - x = \frac{a}{2\sqrt{2}} - \frac{a}{2\sqrt{2}} = 0$$

$\therefore y = x$ [1]

3. Given $x^2 + y^2 - 2x - 3 = 0$

Differentiating both sides w.r.t. x, we get

$2x + 2y\frac{dy}{dx} - 2 = 0$ [1]

$$2y\frac{dy}{dx} = 2 - 2x$$

$$y\frac{dy}{dx} = 1 - x$$

$y\frac{dy}{dx} = 1 - x$ [1]

Slope of x-axis = 0 i.e. $\frac{dy}{dx} = 0$

$\frac{1-x}{y} = 0$ [tangent to x-axis]

$1 - x = 0$

$\Rightarrow x = 1$ [1]

Putting the value of x in (i) we get

$1 + y^2 - 2 - 3 = 0$

$\Rightarrow y^2 = 4$

$\Rightarrow y = \pm 2$

Required points are $(1,2)$ and $(1-2)$. [1]

4. Given $16x^2 + 9y^2 = 145$

Point (x_1, y_1), lies on the given equation.

Thus this point must satisfies the equation.

$16x_1^2 + 9y_1^2 = 145$

Given $x_1 = 2$

$\therefore\ 16(2)^2 + 9y_1^2 = 145$

$\therefore\ 64 + 9y_1^2 = 145$

$\therefore\ 9y_1^2 = 81$

$\therefore\ y_1^2 = 9$

$\therefore\ y_1 = \pm 3$

As $y_1 > 0$

$\Rightarrow y_1 = 3$. [1]

Coordinates of the given point $(x_1, y_1) = (2,3)$.

To find the slope differentiate the equation w.r.t. x

$$32x + 18y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-32x}{18y} = \frac{-16x}{9y}$$

$$\frac{dy}{dx}_{\text{at}(2,3)} = \frac{-16(2)}{9(3)} = \frac{-32}{27}$$

Slope of normal $= \frac{-1}{\frac{dy}{dx}} = \frac{-1}{\frac{-32}{27}} = \frac{27}{32}$ [1]

Equation of tangent at $(2,3)$ with slope $\frac{-32}{27}$

$$y-3=\frac{-32}{27}(x-2)$$

$$27y-81=-32x+64$$

$$32x+27y=145$$ [1]

Equation of normal at $(2,3)$ with slope $\frac{27}{32}$

$$y-3=\frac{27}{32}(x-2)$$

$$32y-96=27x-54$$

$$27x-32y=-42$$ [1]

5. $y=x^3$

On differentiation

$$\frac{dy}{dx}=3x^2$$ [1]

The slope of the tangent at the point (x,y) is,

$$\left[\frac{dy}{dx}\right]_{(x,y)}=3(x)^2$$

When the slope of the tangent is equal to the y-coordinate of the point, then

$$y=3x^2$$ [1]

Also, $y=x^3 \therefore 3x^2=x^3$

$$x^2(x-3)=0$$

$$x=0, x=3$$ [1]

When $x=0$, then $y=0$ and

When $x=3$, then $y=3(3)^2=27$

Hence, the required points are (0, 0) and (3, 27). [1]

6. Let the required point of contact be (x_1, y_1).

Given curve is $y=x^3-11x+5$ ...(i)

$$\therefore \frac{dy}{dx}=3x^2-11$$ [1]

$$\Rightarrow \left[\frac{dy}{dx}\right]_{(x_1,y_1)}=3x^2-11$$

i.e., Slope of tangent at (x1 , y1) to give curve $3x^2-11$

From question

$3x_1^2-11=$ slope of line $y=x-11$ [1]

which is also tangent

$$\therefore\ 3x_1^2-11=1$$

$$\therefore\ x_1^2=4$$

$$\therefore\ x_1=\pm 2$$

Since (x_1, y_1) lie on curve(i)

$$\therefore y_1=x_1^3-11x_1+5$$ [1]

When $x_1=2, y_1=2^3-11\times 2+5=-9$ and when $x_1=-2,\ y_1(-2)^3-11\times(-2)+5=19$ But (-2, 19) does not satisfy the line y = $x-11$

Therefore $(2,-9)$ is required point of curve. [1]

7. The slope of the tangent at to the curve

$$\Rightarrow y'=\left.\frac{b^2x}{a^2y}\right]_{(\sqrt{2}a,b)}=\frac{b^2\sqrt{2}a}{a^2b}=\frac{b\sqrt{2}}{a}$$ [1]

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}-\frac{2yy'}{b^2}=0$$

The equation of the tangent:

Using point slope form

$$y-y_1=m(x-x_1)$$

$$y-b=\frac{b\sqrt{2}}{a}\left(x-\sqrt{2}a\right)$$

$$ay-ab=b\sqrt{2}x-2ab$$

or $b\sqrt{2}x-ay-ab=0$ [1]

Normal:

The slope of the normal

$$= \frac{-1}{\frac{dy}{dx}} = \frac{-1}{b\sqrt{2}} = -\frac{a}{b\sqrt{2}}$$ [1]

Equation of Normal

$$y - b = \frac{-a}{b\sqrt{2}}(x - \sqrt{2}a)$$

$$yb\sqrt{2} - b^2\sqrt{2} = -ax + \sqrt{2}a^2$$

$$\therefore\ ax + b\sqrt{2}y - \sqrt{2}\left(a^2 + b^2\right) = 0$$ [1]

8. $x + 14y + 3 = 0$

$$m = -\frac{1}{14}$$ [1]

Slope of perpendicular line = 14

curve $y = x^3 + 2x - 4$

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3x_1^2 + 2 = 14$$

$$3x_1^2 = 12$$

$$x_1^2 = 4$$

$$x_1 = \pm 2$$

$$x_1 = 2$$

$$y_1 = 8 + 4 - 4 = 8$$ [1]

point $(2,\ 8)$

$$x_1 = -2$$

$$y_1 = -8 - 4 - 4 = -16$$

point $(-2, -16)$

Tangent at $(2, 8)$

$$y - 8 = 14(x - 2) = 14x - 28,$$

$$y = 14x - 20$$ [1]

And at $(-2, -16)$

$$y + 16 = 14(x + 2)$$

$$y + 16 = 14 + 28$$

$$y = 14x + 12$$ [1]

9. Let a point be (x_1, y_1)

$$3x^2 - y^2 = 8$$

$$\therefore\ 6x - 2y.y' = 0$$

$$\therefore\ y' = \frac{3x}{y}$$ [1]

$\therefore$ equation of Tangent, $y - y_1 = \frac{3x_1}{y_1}(x - x_1)$ ...(i)

It is passing through $\left(\frac{4}{3}, 0\right)$.

$$\therefore\ -y_1 = \frac{3x_1}{y_1}\left(\frac{4}{3} - x_1\right)$$ [1]

$$\therefore\ -y_1^2 = 4x_1 - 3x_1^2$$

$$\therefore\ y_1^2 = 3x_1^2 - 4x_1$$

$$\therefore\ 3x_1^2 - 8 = 3x_1^2 - 4x_1$$ [1]

$$\therefore\ x_1 = 2$$

$$\Rightarrow\ 12 - y^2 = 8$$

$$\therefore\ y^2 = 4$$

$$\therefore\ y_1 = \pm 2$$ [1]

Now putting the value of x_1 and y_1 in equation (i) we get, $y = \pm(3x - 4)$ [1]

Hence possible equation of tangents are

$y = -3x + 4$ and $y = 3x - 4$ [1]

[TOPIC 3] Maxima and Minima

Summary

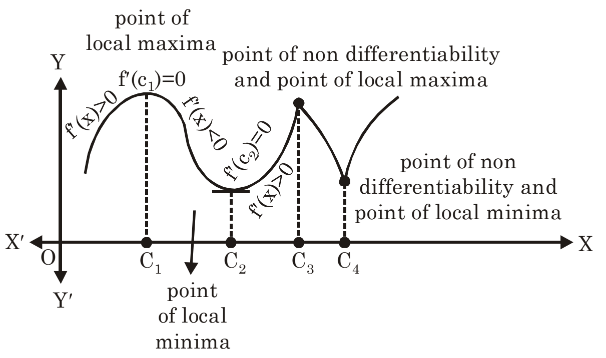
- The maximum value attained by a function is called maxima and the minimum value attained by the function is known as minima.
- Consider $y = f(x)$ be a well-defined function on an interval I, then
  - ➢ f is said to have a maximum value in I, if there exist a point c in I such that $f(c) > f(x)$, $\forall x \in$ I. The value corresponding to f(c) is called as maximum value of x in I and the point c is the maximum value.
  - ➢ f is said to have a minimum value in I, if there exist a point c in I such that $f(c) < f(x)$, $\forall x \in I$. The value corresponding to f(c) is called as minimum value of x in I and the point c is the minimum value.
  - ➢ f is said to have an extreme value in I, if there exist a point c in I such that f(c) is either a maximum value or a minimum value. The value corresponding to f(c) is called as extreme value of x in I and the point c is the extreme point.
- Let f be a function defined on an open interval I. Suppose c ? I be any point. If f has a local maxima or a local minima at x = c, then either f ′(c) = 0 or f is not differentiable at c.

First Derivative Test

Let f be a function defined on an open interval I.

Let f be continuous at a critical point c in I. Then

- If f ′(x) changes sign from positive to negative as x increases through c, i.e., if f′(x) > 0 at every point sufficiently close to and to the left of c, and f ′(x) < 0at every point sufficiently close to and to the right of c, then c is a point of local maxima.
- If f ′(x) changes sign from negative to positive as x increases through c, i.e., if f ′(x) < 0 at every point sufficiently close to and to the left of c, and f ′(x) > 0 at every point sufficiently close to and to the right of c, then c is a point of local minima.
- If f ′(x) does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. In fact, such a point is called point of inflection.



Second Derivative Test

Let f be a function defined on an interval I and c ∈ I. Let f be twice differentiable at c. Then

- x = c is a point of local maxima if f ′(c) = 0 and f ″(c) < 0. The value f (c) is local maximum value of f.
- x = c is a point of local minima if f ′(c) = 0 and f ″(c) > 0 In this case, f (c) is local minimum value of f.
- The test fails if f ′(c) = 0 and f ″(c) = 0. In this case, we go back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflexion.
- **Maximum and Minimum values of a function in a closed interval**

  Let f be a continuous function on an interval I = [a, b]. Then f has the absolute maximum value and f attains it at least once in I. Also, f has the absolute minimum value and attains it at least once in I.

  Let f be a differentiable function on a closed interval I and let c be any interior point of I. Then

  - ➢ f ′(c) = 0 if f attains its absolute maximum value at c.
  - ➢ f ′(c) = 0 if f attains its absolute minimum value at c.

  In view of the above results, we have the following working rule for finding absolute maximum and/ or absolute minimum values of a function in a given closed interval [a, b].

- **Working Rule**
  - ➢ Find all critical points of f in the interval, i.e., find points x where either f ′(x) = 0 or f is not differentiable.
  - ➢ Take the end points of the interval.
  - ➢ At all these points (listed in Step 1 and 2), calculate the values of f
  - ➢ Identify the maximum and minimum values of f out of the values calculated in
  - ➢ This maximum value will be the absolute maximum (greatest) value off and the minimum value will be the absolute minimum (least) value of f.

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 3

4 Marks Questions

1. If the function $f(x) = 2x^3 - 9mx^2 + 12m^2x + 1$, where $m > 0$ attains its maximum and minimum at p and q respectively such that $p^2 = q$, then find the value of m.

[ALL INDIA 2015]

2. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

[ALL INDIA 2011]

6 Marks Questions

3. Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube.

[ALL INDIA 2017]

4. The volume of a cube is increasing at the rate of 9 cm^3/s. How fast is its surface area increasing when the length of an edge is 10 cm ?

[ALL INDIA 2017]

5. If the sum of lengths of the hypotenuse and a side of a right angles triangle is given, show that the area of the triangle is maximum, when the angle between them is $\frac{\pi}{3}$.

[DELHI 2017]

6. Show that the semivertical angle of the cone of the maximum volume and of given slant height is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

[DELHI 2014]

7. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also find the maximum volume in terms of volume of the sphere.

[ALL INDIA 2014]

8. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area,

[DELHI 2011]

9. Show that the height of a closed right circular cylinder of given surface and maximum volume, is equal to the diameter of its base.

[DELHI]

10. Show that semi-vertical angle of a cone of maximum volume and given slant height is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

[ALL INDIA 2016]

11. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is 4r/3, Also show that the maximum volume of the cone is 8/27 of the volume of the sphere.

[DELHI 2014]

Solutions

1. $f(x) = 2x^3 - 9mx^2 + 12m^2x + 1, m > 0$

$f(x) = 6x^2 - 18mx + 12m^2$

$f'(x) = 12x - 18m$ [1]

For Max or Minimum, equate $f'(x)$ to zero,

$f'(x) = 0$

$\therefore\ 6x^2 - 18mx + 12m^2 = 0$

$\therefore\ (x - 2m)(x - m) = 0$

$\therefore\ x = m$ or $2m$

At $x = m$, [1]

$f''(x) = 12m - 18m = -ve$

$\Rightarrow x = m$ is a maxima point,

At $x = 2m$,

$f''(x) = 24m - 18m = +ve$

$\Rightarrow x = m$ is a minima point

$p = m$ and $q = 2m$ [1]

Given

$p^2 = q$

$\therefore\ m^2 = 2m$

$\therefore\ m^2 - 2m = 0$

$\therefore\ m = 0, 2$

$\therefore\ m = 2 \qquad (\because\ m > 0)$ [1]

2. Let the side of the square = x cm

The height of the box = x

The length = $45 - 2x$

The breadth = $24 - 2x$

Volume of the box = $V(x)$

$V = x(45 - 2x)(24 - 2x)$

$V = 2x(45 - 2x)(12 - x)$ [1]

$V'(x) = 2(45 - 2x)(12 - x) - 4x(12 - x) - 2x(45 - 2x)$

$V'(x) = 2[(45 - 2x)(12 - x) - 2x(12 - x) - x(45 - 2x)]$

$V'(x) = 2[540 - 45x - 24x + 2x^2 - 24x + 2x^2 - 45x + 2x^2]$

$V'(x) = 2[540 - 138x + 6x^2]$ [1]

$V'(x) = 12[90 - 23x + x^2]$

$V'(x) = 12(x - 18)(x - 5)$

$V'(x) = 0$

$12(x - 18)(x - 5) = 0 \quad x = 5, 1$

$V''(x) = 12(x - 18) + 12(x - 5)$

$V''(x) = 12(x - 18 + x - 5)$ [1]

At $x = 5$

Now, $V''(5) = 12(10 - 23) < 0$

$\therefore$ By second derivative test, $x = 5$ is the point of maxima. [1]

3. If each side of square base is x and height is h then volume

$V = x^2h$

$\Rightarrow h = \dfrac{V}{x^2}$ [1]

S is surface area then

$S = 4hx + 2x^2 = 4\left(\dfrac{V}{x^2}\right)x + 2x^2$

$\Rightarrow S = \dfrac{4V}{x} + 2x^2$ [1]

Diff. w.r.t. to x

$\dfrac{dS}{dx} = -\dfrac{4V}{x^2} + 4x$ and $\dfrac{d^2S}{dx^2} = +\dfrac{8V}{x^3} + 4$ [1]

Now,

$\dfrac{dS}{dx} = 0$

$\Rightarrow 4x = \dfrac{4V}{x^2}$

$\Rightarrow x^3 = V$

$\Rightarrow x = V^{\frac{1}{3}}$ [1]

Now at $x = V^{\frac{1}{3}}, \dfrac{d^2S}{dx^2} > 0$

$\Rightarrow$ S is minimum when $x = V^{\frac{1}{3}}$ [1]

And

$h = \dfrac{V}{x^2} = \dfrac{V}{V^{\frac{2}{3}}} = V^{\frac{1}{3}}$

$\therefore\ x = h$

$\therefore\ x = h$ indicates that it is a cube. [1]

4. $\frac{dV}{dt} = 9 cm^3 / sec$

$\frac{dA}{dt} = ?$

$l = 10 cm$ [1]

$\frac{dV}{dt} = \frac{d}{dt}(l^3) = 9$

$\therefore \quad 3l^2 \frac{dl}{dt} = 9$ [1]

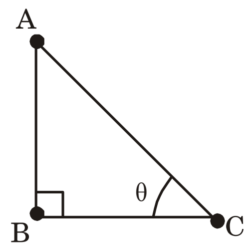
$\frac{dl}{dt} = \frac{3}{l^2}$ ...(i) [1]

Now, $\frac{dA}{dt} = \frac{d}{dt}(6l^2) = 12l\frac{dl}{dt} = 12l \times \frac{3}{l^2}$ From (i)

[2]

$= \frac{36}{l} = \frac{36}{10} = 3.6 cm^2 / sec$ [1]

5. In right angle triangle ΔABC,



Let base, $AB = x$ units and hypotenuse, $AC = y$ units

Let θ be the angle between hypotenuse and base.

[1]

Let $x + y = k$, a constant

$y = k - x$ ...(i)

We know, $BC^2 = AC^2 - AB^2$

(Pythagoras theorem)

$BC^2 = y^2 - x^2$

$BC^2 = (k - x)^2 - x^2$ From (i)

$BC^2 = k^2 + x^2 - 2kx - x^2 = k^2 - 2kx$ (ii) [1]

Area of $\Delta ABC = \frac{1}{2} \times AB \times BC$

$A^2 = \frac{1}{4} \times x^2 \times BC^2$

$A^2 = \frac{1}{4} \times x^2 \times (k^2 - 2kx)$ From (ii) [1]

Let $A^2 = Z$, then $Z = \frac{1}{4} \cdot x^2 (k^2 - 2kx)$

$Z = \frac{k^2 x^2}{4} - \frac{kx^3}{2}$

Differentiating both sides w.r.t. x, we get

$\frac{dZ}{dx} = \frac{k^2}{4}(3x^2) = \frac{k^2 x}{2} - \frac{k3x^2}{2}$ (3)

For maxima or minima $\frac{dZ}{dx} = 0$

$\frac{k}{2}(k - 3x) = 0$

$\frac{kx}{2} = 0$ or $k - 3x = 0$

$x = 0$ (Rejected) $x = \frac{k}{3}$ [1]

Again differentiating both sides w.r.t. x, we get

$\frac{d^2 Z}{dx^2} = \frac{k^2}{2} - \frac{3k}{2}(2x) = \frac{k^2}{2} - 3k(x)$

$\left(\frac{d^2 Z}{dx^2}\right)_{\text{at } x = \frac{k}{3}} = \frac{k^2}{2} - 3k\left(\frac{k}{3}\right)$

$= \frac{k^2}{2} - k^2 = -\frac{k^2}{2} < 0 \, (-ve)$

Z is maximum at. $x = \frac{k}{3}$ [1]

$\therefore A^2$ is maximum at $x = \frac{k}{3}$

From, (1) $y = k - x = k - \frac{k}{3} = \frac{2k}{3}$

$\cos\theta = \frac{AB}{AC} = \frac{x}{y}$

$\cos\theta = \frac{\frac{k}{3}}{\frac{2k}{3}} = \frac{1}{2}$

$\therefore \quad \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ [1]

6. Let r be the radius of base and h be the height of a cone, θ be the semi vertical angle and l be the slant height.

$$l^2 = r^2 + h^2$$

$$l^2 - h^2 = r^2 \quad ...(i) \quad [1]$$

Volume of a cone,

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(l^2 - h^2\right)h \text{ (from equation (i))}$$

$$V = \frac{1}{3}\pi\left(l^2 h - h^3\right) \quad [1]$$

Diffrentiating both sides w.r.t. h, we have

$$\frac{dV}{dh} = \frac{\pi}{3}\left(l^2 - 3h^2\right) \quad ...(ii) \quad [1]$$

For V to be maximum, $\frac{dV}{dh} = 0$

$$\frac{\pi}{3}\left(l^2 - 3h^2\right) = 0 \quad \Rightarrow l^2 - 3h^2 = 0$$

$$\Rightarrow h^2 = \frac{l^2}{3}$$

$$\Rightarrow h = \frac{l}{\sqrt{3}} \quad [1]$$

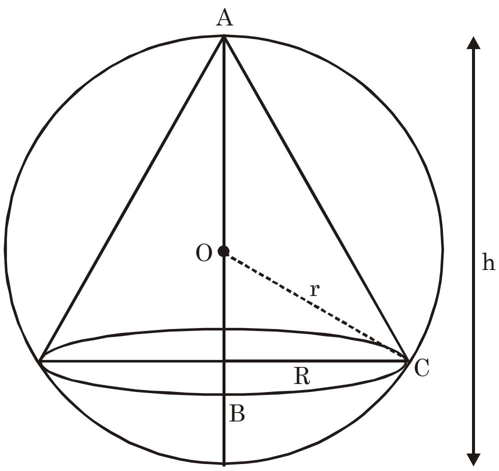
Again differentiating both sides w.r.t. h, we get

$$\frac{d^2V}{dh^2} = \frac{\pi}{3}(-6h) = -2\pi h \quad [1]$$

$$\left[\frac{d^2V}{dh^2}\right]_{at\,h=\frac{l}{\sqrt{3}}} = -2\pi\left(\frac{l}{\sqrt{3}}\right) = -ve$$

Volume is maximum at $h = \frac{l}{\sqrt{3}}$ [1]

7.



Let the altitude of cone be x units

$$OA = OC = r$$

$$OB - (x - r) \quad ...(i) \quad [1]$$

In ΔOBC,

(By Pythagoras theorem)

$$BC = \sqrt{OC^2 - OB^2}$$

$$= \sqrt{r^2 - (x-r)^2}$$

$$= \sqrt{r^2 - x^2 - r^2 + 2xr}$$

$$= \sqrt{2xr - x^2} \quad [1]$$

Volume of cone,

$$V = \frac{1}{3}\pi r^2 x$$

$$= \frac{1}{3}\pi (BC)^2 (AB)$$

$$= \frac{1}{3}\pi\left(2xr - x^2\right)x$$

$$= \frac{\pi}{3}\left(2rx^2 - x^3\right) \quad [1]$$

Differentiating both sides w.r.t. x

$$\frac{dV}{dx} = \frac{\pi}{3}\left(4rx - 3x^2\right)$$

For critical points

$$\frac{dV}{dx} = 0$$

$$\frac{\pi}{3}x(4r - 3x) = 0$$

$$\frac{\pi x}{3} = 0$$

$x = 0$ (not possible)

$$\therefore \quad 4r - 3x = 0$$

$$\therefore \quad x = \frac{4r}{3} \quad [1]$$

$$\frac{d^2V}{dx^2} = \frac{\pi}{3}(4r - 6x)$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=\frac{4r}{3}} = \frac{\pi}{3}\left(4r-6\left(\frac{4r}{3}\right)\right)$$

$$= \frac{\pi}{3}(4r-8r) = \frac{-4r\pi}{3} < 0 \quad [1]$$

Volume of cone is maximum at $x = \frac{4r}{3}$

$$= \frac{\pi}{3}\left[2r.\frac{16r^2}{9} - \frac{64r^3}{27}\right]$$

$$= \frac{\pi}{3}\left[\frac{96r^3 - 64r^3}{27}\right]$$

$$= \frac{\pi}{3}\left[\frac{32r^3}{27}\right]$$

$$= \frac{8\times 4\pi r^3}{27\times 3}$$

$$= \frac{8}{27}\left(\frac{4}{3}\pi r^3\right)$$

Volume of cone $= \frac{8}{27}$ Volume of Sphere. [1]

8. Let rectangle of sides x and y be inscribed in a circle of diameter k.

$$AB^2 + BC^2 = AC^2$$
$$x^2 + y^2 = k^2$$
$$y^2 = k^2 - x^2$$
$$y = \sqrt{k^2 - x^2} \quad [1]$$

Area of rectangle,

$$(A) = xy = x\sqrt{k^2 - x^2}$$

$$(A)^2 = x^2\left(k^2 - x^2\right)$$

$$z = k^2x^2 - x^4$$

$$\frac{dz}{dx} = 2k^2x - 4x^3 \quad [1]$$

For maxima,

$$\frac{dz}{dx} = 0 \Rightarrow 2x\left(k^2 - 2x^2\right) = 0$$

$$\Rightarrow 2x = 0 \quad or \quad k^2 - 2x^2 = 0$$

$$\Rightarrow x = 0 \quad or \quad x^2 = \frac{k^2}{2} \quad [1]$$

(rejecting $x = 0$, because length of rectangle can not be zero.)

$$x = \frac{k}{\sqrt{2}}$$

$$\frac{d^2z}{dx^2} = 2k^2 - 12x^2$$

$$\left[\frac{d^2z}{dx^2}\right]_{at\,x=\frac{k}{\sqrt{2}}} = 2k^2 - 12\left(\frac{k^2}{2}\right) = -4k^2 < 0 \quad [1]$$

Area is maximum at $x = \frac{k}{\sqrt{2}}$

Form (i) , $y = \sqrt{k^2 - \frac{k^2}{2}}$

$$= \sqrt{\frac{k^2}{2}} = \frac{k}{\sqrt{2}} = x$$

Hence area of rectangle is maximum, if it is a square.

$$\cos\theta = \frac{AB}{AC}$$

$$\Rightarrow \cos\theta = \frac{h}{l} \quad \Rightarrow \cos\theta = \frac{\frac{l}{\sqrt{3}}}{l} = \frac{l}{\sqrt{3}}\times\frac{1}{l}$$

$$\cos\theta = \frac{1}{\sqrt{3}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Hence, the semi vertical angle of the cone of the maximum volume is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$. [2]

9. Let r be the radius and h be the height of a closed right circular cylinder

Given: surface area = S

$$2\pi rh + 2\pi r^2 = S$$

$$2\pi rh = S - 2\pi r^2$$

$$\Rightarrow h = \left(\frac{s}{2\pi r} - r\right)$$ [1]

Volume $V = \pi r^2 h = \pi r^2 = \pi r^2\left(\frac{s}{2\pi r} - r\right)$

$$= \frac{sr}{2} - \pi r^3$$ [1]

Differentiating w.r.t, 'r' we get

$$\frac{dv}{dr} = \frac{S}{2} - 3\pi r^2$$ [1]

For stationary point, $\frac{dv}{dr} = 0$

$$\therefore \frac{s}{2} - 3\pi r^2 = 0$$

$$\therefore -3\pi r^2 = -\frac{s}{2}$$

$$\therefore 6\pi r^2 = s$$

$$\therefore 6\pi r^2 = 2\pi rh + 2\pi r^2$$

$$\therefore 6\pi r^2 - 2\pi r^2 = 2\pi rh$$

$$\therefore 4\pi r^2 = 2\pi rh$$

$$\therefore \frac{4\pi r^2}{2\pi r} = h$$

$$2r = h$$ [1]

Diameter of the base of cylinder height of cylinder

$$r = \frac{h}{2}$$

Differentiating (iii),w.r.t. 'r',we get

$$\frac{d^2v}{dr^2} = -6\pi r$$

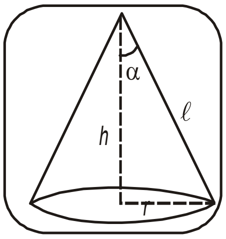
$$\left.\frac{d^2v}{dr^2}\right|_{\text{at } r=\frac{h}{2}} = 6\pi\left(\frac{h}{2}\right) < 0$$ [1]

Volume of closed right circular cylinder is maximum when its height is equal to the diameter of the base.

$$\therefore \cos\theta = \cos\left(\frac{\pi}{3}\right) \Rightarrow \theta = \frac{\pi}{3}$$

Hence, area of triangle is maximum when the angle between base and hypotenuse is $\frac{\pi}{3}$. [1]

10.



Volume $V = \frac{1}{3}\pi r^2 h$

$$V = \frac{1}{3}\pi\left(l^2 - h^2\right)h$$ [1]

For maxima and minima

$$\frac{dv}{dh} = 0$$

$$\frac{1}{3}\pi\left(l^2 - 3h^2\right) = 0$$

$$\Rightarrow l^2 = 3h^2$$

$$\Rightarrow h = \frac{\ell}{\sqrt{3}}$$ [1]

And

$$\frac{d^2y}{dh^2} = \frac{1}{3}\pi(0 - 6h)$$

$$= -2\pi h$$

$$= -2\pi\left[\frac{\ell}{\sqrt{3}}\right] < 0$$ [1]

So at $h = \frac{\ell}{\sqrt{3}}$ volume of cone is maximum [1]

and semivertical angle α as:

$$\cos\alpha = \frac{h}{l}$$

$$\cos\alpha = \frac{\frac{l}{\sqrt{3}}}{l}$$

$$\cos\alpha = \frac{1}{\sqrt{3}}$$

$$\therefore \alpha = \cos^{-1}\left[\frac{1}{\sqrt{3}}\right]$$ [2]

11. Let R and h be the radius and height of the cone.
r be the radius of sphere
In ΔABC, $AC = h - r$

$\therefore (h-r)^2 + R^2 = r^2$ {Pythagoras Theorem}

$\Rightarrow R^2 = r^2 - (h-r)^2$ [1]

Volume of cone : $V = \frac{1}{3}\pi R^2 h$

Or $V = \frac{1}{3}\pi\left(r^2 - (h-r)^2\right)h$

$V = \frac{1}{3}\pi(r^2 - h^2 - r^2 + 2hr)h$

$V = \frac{1}{3}\pi(2h^2 r - h^3)$ [1]

For maxima or minima volume $\frac{dV}{dh} = 0$

Now, $\frac{dV}{dh} = \frac{1}{3}\pi[4hr - 3h^2]$

equating to zero $\frac{dV}{dh}$

We get $4\text{hr} = 3h^2$

$\therefore h = \frac{4r}{3}$ [1]

$\frac{d^2V}{dh^2} = \frac{1}{3}\pi[4r - 6h]$

Putting $h = \frac{4r}{3}$

$\frac{d^2V}{dh^2} = \frac{1}{3}\pi\left[4r - \frac{6.4r}{3}\right]$

$= -\frac{1}{3}\pi[4r]$

Which is less than zero, therefore

$h = \frac{4r}{3}$ is a maxima [1]

and the Volume of the cone at $h = \frac{4r}{3}$ will be maximum.

$V = \frac{1}{3}\pi R^2 h$

$= \frac{1}{3}\pi\left[r^2 - (h-r)^2\right]h$

$= \frac{1}{3}\pi\left[r^2 - \left(\frac{4r}{3} - r\right)^2\right]\left[\frac{4r}{3}\right]$

$= \frac{1}{3}\pi\left[r^2 - \left(\frac{4r}{3} - r\right)^2\right]\left[\frac{4r}{3}\right]$ [1]

$= \frac{1}{3}\pi\left[\frac{8r^2}{9}\right]\left[\frac{4r}{3}\right]$

$= \frac{8}{27}\left(\frac{4\pi r^3}{3}\right)$

$= \frac{8}{27}$ [1]

Value Based Questions

PREVIOUS YEARS'

EXAMINATION QUESTIONS

4 Marks Questions

1. The money to be spent for welfare of the employees of a firm is proportional to the rate of change of it's total revenue (marginal revenue). If the total revenue (in rupees) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, find the marginal revenue, when , and write which value does the equation indicate.

[ALL INDIA 2013]

2. The amount of pollution content added in a air in a city to x-diesel vehicles is given by $P(x) = 0.005x^3 + 0.02x^2 + 30x$ find the marginal increase in pollution content when 3 diesel vehicles are added and write which value is indicated in the above question.

[DELHI 2013]

3. The total cost $C(x)$ in associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.

[DELHI 2018]

4. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. If the cost is to be borne by nearby settled lower income families, for whom water will be provided, what kind of value is hidden in this question?

[DELHI 2018]

Solutions

1. $R(x) = 3x^2 + 36x + 5$

$$MR = \frac{dR}{dx} = 6x^2 + 36$$

When $x = 5$

MR = 30 + 36 = 66 [2]

Value : Value depicted from above question is care and affinity for employees. [2]

2. $P(x) = 0.005x^3 + 0.02x^2 + 30x$

Differentiating both sides w.r.t.x, we get

The marginal increase in pollution content when $x = 3$

$$p'(x) = 0.015x^2 + 0.04x + 30$$ [1]

$$p'(3) = 0.015(3)^2 + 0.04(3) + 30$$

$$= 0.135 + 0.12 + 30 = \mathbf{30.255}$$ [2]

Pollution caused by diesel vehicles is a major cause of concern. Smoke emitted by vehicles leads to global warming. Thus we must make all efforts to save our environment by going green and switching to alternate fuels like CNG. [1]

3. Given $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$

To find the marginal cost we need to differentiate the cost function with respect to output.

$$\therefore \text{Marginal cost} = \frac{dC}{dx}$$

$$= 0.005(3x^2) - 0.02(2x) + 30$$

$$= 0.015x^2 - 0.04x + 30$$ [1]

Given that $x = 3$

$$\text{Marginal cost} = 0.015(3^2) - 0.04(3) + 30$$

$$= 0.015(9) - 0.12 + 30$$

$$= 0.135 - 0.12 + 30$$

$$= \mathbf{30.015}$$ [2]

Hence, the marginal cost is Rs 30.015 when 3 units are produced. [1]

4. Given an open tank with square base

Length = Breadth

Let length of tank = x units

Then breadth of tank = x units

Let height of tank = y units

Volume of Cuboid = $l \times b \times h$

Volume of tank $V = x \times x \times y = x^2 y$ [1]

$$y = \frac{V}{x^2} \quad \ldots(\text{i})$$

Let the cost of the metal sheet used for tank be Rs C per square unit

Total surface area of open cuboid

$$= lb + 2h(l + b)$$

Total cost $T = C(x \times x + 2y(x + x))$

$$= C(x^2 + 4xy)$$

From (1)

$$T = C\left(x^2 + 4x\left(\frac{V}{x^2}\right)\right)$$

$$= Cx^2 + \frac{4CV}{x}$$

Differentiating both sides w.r.t. x we get,

$$\frac{dT}{dx} = 2Cx - \frac{4CV}{x^2}$$

Checking critical point

$$0 = 2Cx - \frac{4CV}{x^2}$$

$$\frac{4CV}{x^2} = 2Cx$$

$$4CV = 2Cx^3$$

$$2V = x^3$$

$$x = \sqrt[3]{2V} \quad \ldots(\text{ii})$$ [1]

Differentiating again $\frac{dT}{dx} = 2Cx - \frac{4CV}{x^2}$ w.r.t. x we get

$$\frac{d^2T}{dx^2} = 2C + \frac{8CV}{x^3}$$

From 2 we get

$$\frac{d^2T}{dx^2} = 2C + \frac{8CV}{\left(\sqrt[3]{2V}\right)^3}$$

$$\frac{d^2T}{dx^2} = 2C + \frac{8CV}{2V} = 2C + 4C = 6C$$

As $6C > 0$ [1]

Thus function is minimum

Total cost is minimum at $x = \sqrt[3]{2V}$

From (i) we get

$$y = \frac{V}{(2V)^{\frac{2}{3}}} = \frac{V^{\frac{1}{3}}}{2^{\frac{2}{3}}} = \frac{V^{\frac{1}{3}} \times 2^{\frac{1}{3}}}{2^{\frac{2}{3}} \times 2^{\frac{1}{3}}} = \frac{1}{2}(2V)^{\frac{1}{3}}$$

From (ii) we get,

$$y = \frac{1}{2}x$$ [1]

Base is directly proportional to height.

Smart Notes ....

Smart Notes ....

CHAPTER 7

Integrals

Chapter Analysis of Previous Three Years' Board Exams

Number of Questions Asked in Exams

Topics	2018		2017		2016	
	Delhi	**All India**	**Delhi**	**All India**	**Delhi**	**All India**
Indefinite Integral	2, 4 marks	2, 4 marks	2, 4 marks	1, 2, 4 marks	4, 4 marks	4, 4 marks
Definite Integral	6 marks	6 marks	1, 4 marks	4 marks	4 marks	4 marks

[TOPIC 1] Indefinite Integrals

Summary

- Integration is the inverse of differentiation. Instead of differentiating a function, we will be given the derivative of a function and we would be asked to find its primitive function. Such a process is called integration or anti differentiation.

  If $\frac{d}{dx}F(x) = f(x)$. Then we write $\int f(x)dx = F(x) + C$.

  These integrals are called indefinite integrals or general integrals and C is called the constant of integration.
- The integral of a function are unique upto an additive constant, i.e. any two integrals of a function differ by a constant.
- When a polynomial function P is integrated, the result is a polynomial whose degree is one more than that of P.
- Geometrically, the indefinite integral of a function represents a family of curves placed parallel to each other having parallel tangents at the points of intersection of the curves of the family with the lines perpendicular to the axisrepresenting the variable of integration.
- Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent.
- Some properties of indefinite integral are as follows:
  - ➢ $\int [f(x) + g(x)]\,dx = \int f(x)dx + \int g(x)dx$
  - ➢ For any real number a, $\int af(x)dx = a\int f(x)dx$
  - ➢ Properties (i) and (ii) can be generalised to a finite number of functions $f_1, f_2, f_3, ...f_n$ and the real numbers, $a_1, a_2, a_3, ...a_n$ giving

    $$\int a_1 f_1(x)\mathrm{d}x + a_2 f_2(x)\mathrm{d}x + ... + a_n f_n(x)dx$$

    $$= a_1\int f_1(\mathrm{x})\mathrm{dx} + \mathrm{a}_2\int f_2(x)\mathrm{dx} + ...\mathrm{a}_n\int f_n(x)dx.$$
- Some basic integrals are as follows:
  - ➢ $\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad , n \neq -1$
  - ➢ $\int dx = x + C$
  - ➢ $\int \cos x dx = \sin x + C$
  - ➢ $\int \sin x dx = -\cos x + C$
  - ➢ $\int \sec^2 x dx = \tan x + C$
  - ➢ $\int \mathrm{cosec}^2 x dx = -\cot x + C$
  - ➢ $\int \sec x \tan x dx = \sec x + C$
  - ➢ $\int \mathrm{cosec}\, x \cot x\, dx = -\mathrm{cosec}\, x + C$
  - ➢ $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
  - ➢ $\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$
  - ➢ $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
  - ➢ $\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$
  - ➢ $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$
  - ➢ $\int \frac{dx}{x\sqrt{x^2-1}} = -\mathrm{cosec}^{-1} x + C$
  - ➢ $\int e^x dx = e^x + C$
  - ➢ $\int \frac{1}{x} dx = \log|x| + C$
  - ➢ $\int a^x dx = \frac{a^x}{\log a} + C$
- **Integration by substitution:**

  The method in which we change the variable to some other variable is called the method of substitution. It often reduces an integral to one of the fundamental integrals.

  The integral $\int f(x)dx$ can be substituted into another form by changing the independent variable x to t by substituting $x = g(t)$.

Consider, $A = \int f(x)dx$

Put $x = g(t)$, therefore $\frac{dx}{dt} = g'(t)$.

$\Rightarrow dx = g'(t)dt$

Thus, $A = \int f(x)\,dx = \int f(g(t))\,g'(t)\,dt$

Using substitution method we obtain the following standard integrals.

- $\int \tan x dx = \log|\sec x| + C$
- $\int \cot x dx = \log|\sin x| + C$
- $\int \sec x dx = \log|\sec x + \tan x| + C$
- $\int \text{cosec}\, x dx = \log|\text{cosec}\, x - \cot x| + C$

- **Integrals of some particular functions are as follows:**
  - $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$
  - $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$
  - $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
  - $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$
  - $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$
  - $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$

- **Integration by Partial Fractions:**
  A rational function is defined as the ratio of two polynomials in the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x and $Q(x) \neq 0$. If $\frac{P(x)}{Q(x)}$ is improper, then $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$ where T(x) is a polynomial in x and $\frac{P_1(x)}{Q(x)}$ is a proper rational function. Assume we want to evaluate $\int \frac{P(x)}{Q(x)}dx$, where $\frac{P(x)}{Q(x)}$ is a proper rational function. It is possible to write the integrand as a sum of simpler rational functions by partial fraction decomposition as follows:
  - $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$, $a \neq b$
  - $\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$
  - $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
  - $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
  - $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$, where

    $x^2 + bx + c$ cannot be factorized further.

- **Integration by parts:**
  If $f(x)$ and $g(x)$ are the two functions then,

  $\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left[f'(x)\int g(x)dx\right]dx$,

  where $f(x)$ is the first function and $g(x)$ is the second function. It can be stated as follows:**"The integration of the product of two functions = (First function) x (integral of the second function) – Integral of [(differential coefficient of the first function) x (integral of the second function)]"**

- Integral of the type $\int e^x\left[f(x) + f'(x)\right]dx = e^x f(x) + C$
- Some special types of integrals are as follows:
  - $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$
  - $\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$

➢ $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

➢ Integrals of the types $\int\frac{dx}{ax^2+bx+c}$ or $\int\frac{dx}{\sqrt{ax^2+bx+c}}$ can be transformed into standard form by expressing ax^2+bx+c $=a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]=a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$

➢ Integrals of the types $\int\frac{px+qdx}{ax^2+bx+c}$ or $\int\frac{px+qdx}{\sqrt{ax^2+bx+c}}$ can be transformed into standard form by expressing $px+q$ $=A\frac{d}{dx}(ax^2+bx+c)+B$ $=A(2ax+b)+B$, where A and B are determined by comparing coefficients on both sides.

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

1 Mark Questions

1. Write the anti derivative of $\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)$.

[DELHI 2014]

2. Evaluate $\int(1-x)\sqrt{x}dx.$

[ALL INDIA 2012]

3. Write the value of $\int\frac{dx}{x^2+16}$.

[DELHI 2011]

4. Write the value of $\int\sec x(\sec x+\tan x)$

[DELHI 2011]

5. Evaluate $\int\frac{(1+\log x)^2}{x}dx$

[ALL INDIA 2011]

6. Find : $\int\frac{\sin^2 x-\cos^2 x}{sinxcosx}dx$

[ALL INDIA 2017]

2 Marks Questions

7. Find $\int\frac{dx}{x^2+4x+8}$

[DELHI 2017]

8. Find : $\int\frac{dx}{5-8x-x^2}$

[ALL INDIA 2017]

9. $\int\frac{\cos 2x+2\sin^2 x}{\cos^2 x}dx$

[DELHI 2018]

4 Marks Questions

10. Evaluate: $\int\frac{\sin(x-a)}{\sin(x+a)}dx$

[DELHI 2013]

11. Integrate the following w.r.t. x: $\frac{x^2-3x+1}{\sqrt{1-x^2}}$.

[DELHI 2015]

12. Evaluate: $\int\frac{x^2}{(x^2+4)(x^2+9)}dx$

[DELHI 2013]

13. Find $\int\frac{2x}{(x^2+1)(x^2+2)^2}dx$

[DELHI 2017]

14. Find $\int \frac{(3\sin\theta-2)\cos\theta}{5-\cos^2\theta-4\sin\theta}\,d\theta$

[DELHI 2016]

15. Find $\int \frac{\sqrt{x}}{\sqrt{a^3-x^3}}\,dx$

[DELHI 2016]

16. Evaluate: $\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x\cos^2 x}\,dx$

[DELHI 2014]

17. Evaluate $\int (x-3)\sqrt{x^2+3x-18}\,dx$

[DELHI 2014]

18. Evaluate: $\int \sin x\sin 2x\sin 3x\,dx$

[ALL INDIA 2012]

19. Evaluate: $\int \frac{2}{(1-x)(1+x^2)}dx$

[DELHI 2012]

20. Evaluate $\int \frac{2x}{\sqrt{(x^2+1)(x^2+3)}}dx$

[DELHI 2011]

21. Evaluate $\int e^{2x}\sin x\,dx$

[ALL INDIA 2011]

22. Evaluate $\int \frac{x+3}{\sqrt{x^2-2x-5}}\,dx$

[ALL INDIA 2011]

23. Evaluate: $\int \frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}\,dx$

[ALL INDIA 2013]

24. Evaluate : $\int \frac{x+2}{\sqrt{x^2+2x+3}}\,dx$

[DELHI 2013]

25. Evaluate: $\int \frac{dx}{x(x^5+3)}$

[ALL INDIA 2013]

26. Evaluate: $\int \frac{x+2}{\sqrt{x^2+5x+6}}dx$

[ALL INDIA 2014]

27. Find : $\int \frac{x^3-1}{x(x^2+1)}\,dx$

[ALL INDIA 2015]

28. Find : $\int (x+3)\sqrt{3-4x-x^2}\,dx$

[ALL INDIA 2016]

29. Find : $\int \frac{(2x-5)\,e^{2x}}{(2x-3)^3}\,dx$

[ALL INDIA 2016]

30. Find : $\int \frac{x^2+x+1}{(x^2+1)(x+2)}\,dx$

[ALL INDIA 2016]

31. $I=\int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}\,d\theta$

[ALL INDIA 2017]

32. Find : $\int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}\,dx$

[DELHI 2018]

33. Find: $\int \frac{dx}{\sin x+\sin 2x}$

[DELHI 2015]

6 Marks Questions

34. $\int \frac{1}{\cos^4 x+\sin^4 x}\,dx$

[ALL INDIA 2014]

35. Evaluate: $\int (\sqrt{\cot x}+\sqrt{\tan x})\,dx$

[ALL INDIA 2015]

Solutions

1. We have $\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)$

$= \int\left(3x^{1/2}+x^{-1/2}\right)dx$

$= 3.\frac{2}{3}x^{3/2}+\frac{2}{1}x^{1/2}+c$

$= 2x\sqrt{x}+2\sqrt{x}+c$

$= 2\sqrt{x}(x+1)+c$ [1]

2. Given: $\int(1-x)\sqrt{x}dx$

$= \int\left(x^{1/2}-x^{3/2}\right)dx$

$= \frac{2}{3}x^{3/2}-\frac{2}{5}x^{5/2}+C$ [1]

3. Given $\int\frac{dx}{x^2+16}=\int\frac{dx}{x^2+4^2}$

Using: $\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+C$

$= \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)+C$ [1]

4. Given $\int\sec x(\sec x+\tan x)\,dx$

$\int\left(\sec^2 x+\sec x\tan x\right)dx$

$= \tan x+\sec x+C$ [1]

5. $\int\frac{(1+\log x)^2}{x}dx$

Let $1+\log x = t$

$\frac{1}{x}dx = dt$ [½]

Put the value of $\frac{1}{x}dx$ in the expression

$= \int t^2 dt$

$= \frac{t^3}{3}+C$

$= \frac{(1+\log x)^3}{3}+C$ [½]

6. $= \int\frac{\sin^2 x-\cos^2 x}{sinxcosx}dx$

$= 2\int\frac{-cos^2 x}{sin^2 x}dx$

Let $t = \sin 2x$

$dt = 2\cos 2x dx$ [½]

$\frac{dt}{2} = \cos 2x dx$

$= \frac{-2}{2}\int\frac{dt}{t}$

$= \frac{-2\log|t|}{2}$

$= \frac{-2\log|sin2x|}{2}+C$

$= -\log|sin2x|+C$ [½]

7. Given $\int\frac{dx}{x^2+4x+8}$

$= \int\frac{dx}{x^2+4x+2^2-2^2+8}$

$= \int\frac{dx}{(x+2)^2+2^2}$ [1]

Using $\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

$= \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right)+C$ [1]

8. $\int\frac{dx}{5-8x-x^2}$

$= \int\frac{dx}{\left\{(x+4)^2-21\right\}}$ [1]

$\int\frac{dx}{\left(\sqrt{21}\right)^2-(x+4)^2}$

$\frac{1}{2\sqrt{21}}\log\left|\frac{\sqrt{21}+(x+4)}{\sqrt{21}-(x+4)}\right|+C$ [1]

9. $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$

$= \int \frac{\cos 2x + (1 - \cos 2x)}{\cos^2 x} dx$ [1]

$= \int \frac{1}{\cos^2 x} dx$

$= \int \sec^2 x dx = \tan x + C$ [1]

10. Given: $\int \frac{\sin(x-a)}{\sin(x+a)} dx$

$[\sin(A-B) = (\sin A \cos B - \cos A \sin B)]$

$= \int \frac{\sin(x+a-a-a)}{\sin(x+a)} dx = \int \frac{\sin[(x+a)-2a]}{\sin(x-a)}$ [1]

$= \int \left(\frac{\sin(x+a)\cos 2a}{\sin(x+a)} - \frac{\cos(x+a)\sin 2a}{\sin(x+a)} \right)$ [1]

$= \int (\cos 2a - \sin 2a \cot(x+a)) dx$ [1]

$= x.\cos 2a - \sin 2a \log|\sin(x+a)| + C$ [1]

11. $\int \frac{x^2 - 3x + 1}{\sqrt{1-x^2}} dx$

$= -\int \frac{-x^2 + 3x - 1}{\sqrt{1-x^2}} dx$

$= -\int \left(\frac{(-x^2+1) + 3x - 1 - 1}{\sqrt{1-x^2}} \right) dx$

$= -\int \left(\frac{(1-x^2)}{\sqrt{1-x^2}} + \frac{3x}{\sqrt{1-x^2}} - \frac{2}{\sqrt{1-x^2}} \right) dx$ [1]

$= -\int \sqrt{1-x^2} dx - 3\int \frac{x}{\sqrt{1-x^2}} dx + 2\int \frac{1}{\sqrt{1-x^2}} dx$

Let $p = 1 - x^2, dp = -2x dx, \frac{dp}{-2} = x dx$ [1]

$= \frac{1}{2}\left[x\sqrt{1-x^2} + 1\sin^{-1}\frac{x}{1} \right] + \frac{3}{2}\int \frac{dp}{\sqrt{p}} + 2\sin^{-1}\frac{x}{1}$

$= \frac{-x}{2}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1} x + \frac{3}{2}\int p^{-\frac{1}{2}} dp + 2\sin^{-1} x$ [1]

$= \frac{-x}{2}\sqrt{1-x^2} + \frac{3}{2}\sin^{-1} x + \frac{3}{2}.\frac{2}{1} p^{\frac{1}{2}} + c$

$= \frac{-x}{2}\sqrt{1-x^2} + \frac{3}{2}\sin^{-1} x + 3\sqrt{1-x^2} + c$

$= \sqrt{1-x^2}\left(3 - \frac{x}{2}\right) + \frac{3}{2}\sin^{-1} x + c$

$= \frac{(6-x)}{2}\sqrt{1-x^2} + \frac{3}{2}\sin^{-1} x + c$ [1]

12. Given: *To find* : $\int \frac{x^2}{(x^2+4)(x^2+9)} dx$

$$\frac{x^2}{(x^2+4)(x^2+9)} = \frac{y}{(y+4)(y+9)}$$

Let $\frac{y}{(y+4)(y+9)} = \frac{A}{(y+4)} + \frac{B}{(y+9)}$ ...(i) [1]

$y = A(y+9) + B(y+4)$

Comparing the coefficients of y *and constant* term.

$1 = A + B$ and $0 = 9A + 4B$

$\Rightarrow 1 - B = A \quad 0 = 9(1-B) + 4B$

$0 = 9 - 9B + 4B$

$5B = 9$

$B = \frac{9}{5}$

$A = 1 - \frac{9}{5} = \frac{-4}{5}$ [1]

Putting the values of A and B in(*i*)

$\left[\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} C \right]$

$$\frac{y}{(y+4)(y+9)} = \frac{-\frac{4}{5}}{y+4} + \frac{\frac{9}{5}}{y+9}$$ [1]

$$\int \frac{x^2}{(x^2+4)(x^2+9)} = \int \left(\frac{-\frac{4}{5}}{x^2+2^2} + \frac{\frac{9}{5}}{x^2+3^2} \right) dx$$

$$= \frac{-4}{5} \times \frac{1}{2} \tan^{-1}\frac{x}{2} + \frac{9}{5} \times \frac{1}{3} \tan^{-1}\frac{x}{3} + C$$

$$= -\frac{2}{5} \tan^{-1}\frac{x}{2} + \frac{3}{5} \tan^{-1}\frac{x}{3} + C \quad [1]$$

13. Given $\int \frac{2x}{(x^2+1)(x^2+2)^2} dx$

Let $I = \int \frac{2x}{(x^2+1)(x^2+2)^2} dx$

Let $x^2 = t$ [1]

$$2x = \frac{dt}{dx}$$

$$2x dx = dt$$

Substituting in above expression we get

$$I = \int \frac{dt}{(t+1)(t+2)^2}$$

Let

$$\frac{1}{(t+1)(t+2)^2} = \frac{A}{(t+1)} + \frac{B}{(t+2)} + \frac{C}{(t+2)^2} \quad ...(1) \quad [1]$$

$$\frac{1}{(t+1)(t+2)^2} = \frac{A(t+2)^2 + B(t+1)(t+2) + C(t+1)}{(t+1)(t+2)^2}$$

$$1 = A(t+2)^2 + B(t+1)(t+2) + C(t+1)$$

$$1 = A(t^2+4+4t) + B(t^2+3t+2) + C(t+1)$$

$$1 = At^2 + 4A + 4tA + Bt^2 + 3tB + 2B + Ct + C$$

Comparing the coefficients of t, t^2 and constant.

$$0 = A + B$$
$$A = -B \quad ...(2)$$

$$0 = 4A + 3B + C$$

From (2) we get

$$0 = 4A - 3A + C$$

$$-C = A \quad ...(3)$$

$$1 = 4A + 2B + C$$

From (2) and (3) we get

$$1 = 4A - 2A - A$$
$$1 = A$$

From (2) and (3)

$$A = 1, B = -1, C = -1$$

From (1) we get

$$\frac{1}{(t+1)(t+2)^2} = \frac{1}{(t+1)} + \frac{-1}{(t+2)} + \frac{-1}{(t+2)^2} \quad [1]$$

Integrating both sides we get

$$\int \frac{1}{(t+1)(t+2)^2} dt = \int \left[\frac{1}{(t+1)} + \frac{-1}{(t+2)} + \frac{-1}{(t+2)^2} \right] dt$$

$$= \log|t+1| - \log|t+2| - \frac{(t+2)^{-1}}{-1} + C$$

$$I = \log|t+1| - \log|t+2| + \frac{1}{(t+2)} + C$$

Substituting $t = x^2$

$$I = \log|x^2+1| - \log|x^2+2| + \frac{1}{(x^2+2)} + C \quad [1]$$

14. Let $I = \int \frac{(3\sin\theta - 2)\cos\theta}{5 - \cos^2\theta - 4\sin\theta} d\theta$

$$= \int \frac{(3\sin\theta - 2)\cos\theta d\theta}{5 - (1 - \sin^2\theta) - 4\sin\theta}$$

$$= \int \frac{(3\sin\theta - 2)\cos\theta d\theta}{\sin^2\theta - 4\sin\theta + 4}$$

$$= \int \frac{(3p - 2}{p^2 - 4p + 4} dp \left[Let\ p = \sin\theta, dp = \cos\theta d\theta \right] \quad [1]$$

$$= \int \frac{(3p-2)}{(p-2)^2} dp$$

Let $\frac{(3p-2)}{(p-2)^2} = \frac{A}{p-2} + \frac{B}{(p-2)^2}$

$3p-2 = A(p-2)+B$

Comparing the coefficients of p and constant terms, we get

$A = 3$

$B-2A = -2 \Rightarrow B-2(3) = -2 \Rightarrow B = 4$ [1]

$$\frac{3p-2}{(p-2)^2} = \frac{3}{p-2} + \frac{4}{(p-2)^2}$$

$$\int \frac{3p-2}{(p-2)^2}\,dp = \int\left(\frac{3}{p-2} + \frac{4}{(p-2)^2}\right)dp$$ [1]

$$I = 3\log|p-2| + 4\left(\frac{-1}{p-2}\right) + c$$

$$= 3\log|\sin\theta - 2| - \frac{4}{\sin\theta - 2} + c$$ [1]

15. $\int \frac{x^{\frac{1}{2}}}{\sqrt{a^3 - (x^{\frac{3}{2}})^2}}\,dx$

$\because$ *Let* $p = x^{\frac{3}{2}}$ [1]

$$dp = \frac{3}{2}x^{\frac{1}{2}}\,dx$$

$$\frac{2}{3}dp = x^{\frac{1}{2}}\,dx$$

$$= \frac{2}{3}\int \frac{dp}{\sqrt{\left(a^{\frac{3}{2}}\right)^2 - (p)^2}}$$ [1]

$$= \frac{2}{3}\sin^{-1}\left(\frac{p}{a^{\frac{3}{2}}}\right) + c \quad \left[\because \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c\right]$$ [1]

$$\frac{2}{3}\sin^{-1}\frac{x^{\frac{3}{2}}}{a^{\frac{3}{2}}} + c \text{ or } \frac{2}{3}\sin^{-1}\left(\sqrt{\frac{x^3}{a^3}}\right) + c$$ [1]

16. $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x}\,dx$

$$= \int \frac{\left(\sin^2 x\right)^3 + \left(\cos^2 x\right)^3}{\sin^2 x \cos^2 x}\,dx$$ [1]

Using identity $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

$$\int \frac{\left(\sin^2 x + \cos^2 x\right)^3 - 3\sin^2 x \cos^2 x(\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x}\,dx$$ [1]

$$= \int \frac{1 - 3\sin^2 x\cos^2 x}{\sin^2 x \cos^2 x}\,dx \left[\because \sin^2 x + \cos^2 x = 1\right]$$

$$= \int \frac{1}{\sin^2 x \cos^2 x}\,dx - \int \frac{3\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x}\,dx$$

$$= \int\left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\right)dx - \int 3\,dx$$ [1]

$$= \int\left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x}\right)dx - \int 3\,dx$$

$$= \int(\sec^2 x + \operatorname{cosec}^2 x) - \int 3\,dx$$

$$= \tan x - \cot x - 3x + c$$ [1]

17. We have, $\int (x-3)\sqrt{x^2+3x-18}\,dx$

$$= \frac{1}{2}\int 2(x-3)\sqrt{x^2+3x-18}\,dx$$

$$= \frac{1}{2}\int (2x-6)\sqrt{x^2+3x-18}\,dx$$

$$= \frac{1}{2}\int \left[(2x+3)-3-6\right]\sqrt{x^2+3x-18}\,dx$$

$$= \frac{1}{2}\int (2x+3)\sqrt{x^2+3x-18}\,dx - \frac{9}{2}\int\sqrt{x^2+3x-18}\,dx$$ [1]

Let $p = x^2+3x-15$

$\therefore dp = (2x+3)\,dx$ [1]

$$= \frac{1}{2}\int \sqrt{p}\, dp - \frac{9}{2}\int \sqrt{x^2 + 3x + \left(\frac{3}{2}\right)^2 - 18 - \left(\frac{3}{2}\right)^2}\, dx$$

$$= \frac{1}{2}\int \sqrt{p}\, dp - \frac{9}{2}\int \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\, dx$$

$$= \frac{1}{2} \times \frac{2}{3} p^{3/2} - \frac{9}{2} \times \frac{1}{2}\left[\left(x + \frac{3}{2}\right)\sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\right.$$

$$\left. - \frac{81}{4}\log\left|x + \frac{3}{2} + \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\right|\right] \quad [1]$$

$$= \frac{1}{3}\left(x^2 + 3x - 18\right)^{3/2} - \frac{9}{4}\left[\left(\frac{2x+3}{2}\right)\sqrt{\left(x^2 + 3x - 18\right)}\right.$$

$$\left.\left|\frac{2x+3}{2} + \sqrt{\left(x^2 + 3x - 18\right)}\right|\right] + C$$

$$= \frac{1}{3}\left(x^2 + 3x - 18\right)^{3/2} - \frac{9}{8}(2x + 3)\sqrt{\left(x^2 + 3x - 18\right)}$$

$$+ \frac{729}{16}\log\left|\frac{2x+3}{2} + \sqrt{\left(x^2 + 3x - 18\right)}\right| + C \quad [1]$$

18. Given $\int \sin x \sin 2x \sin 3x \, dx$

$$\left[2\sin A \sin B = \cos(A - B) - \cos(A + B)\right]$$

$$= \frac{1}{2}\int (2\sin 3x.\sin x).\sin 2x \, dx \quad [1]$$

$$= \frac{1}{2}\int (\cos 2x - \cos 4x).\sin 2x \, dx$$

$$= \frac{1}{2x2}\int 2(\cos 2x \sin 2x - \cos 4x \sin 2x)dx \quad [1]$$

$$\begin{bmatrix} 2\sin A \cos A = \sin 2A \\ 2\cos A \sin B = \sin(A + B) - \sin(A - B) \end{bmatrix}$$

$$= \frac{1}{4}\left[\int 2\sin 2x \cos 2x \, dx - \int 2\cos 4x \sin 2x \, dx\right] \quad [1]$$

$$= \frac{1}{4}\left[\int \sin 4x \, dx - \int (\sin 6x - \sin 2x)\, dx\right]$$

$$= \frac{1}{4}\left[\frac{-\cos 4x}{4} + \frac{\cos 6x}{6} - \frac{\cos 2x}{2}\right] + C$$

$$= \frac{1}{4}\left[\frac{\cos 6x}{6} - \frac{\cos 4x}{4} - \frac{\cos 2x}{2}\right] + C \quad [1]$$

19. Given $\dfrac{2}{(1+x)\left(1+x^2\right)} = \dfrac{A}{1-x} + \dfrac{Bx + C}{1 + x^2}$

$$2 = A\left(1 + x^2\right) + (Bx + C)(1 - x)$$

$$2 = A\left(1 + x^2\right) + Bx(1 - x) + C(1 - x) \quad [1]$$

Comparing the coefficients of x^2, x constant term repectively

$0 = A - B$

$\Rightarrow B = A$

$0 = B - C$

$\Rightarrow B = C$

$2 = A + C$

$\Rightarrow 2 = A + A$

$\Rightarrow A = 1$

$A = B = C = 1$ [1]

Putting the value of A,B and C in(i)

Let $p = 1 + x^2$

$\dfrac{dp}{dx} = 2x$

$dp = 2xdx$

$\dfrac{dp}{2} = xdx$

$$= \frac{2}{(1-x)\left(1+x^2\right)} = \frac{1}{1-x} + \frac{x+1}{1+x^2} = \int \frac{2}{(1-x)\left(1+x^2\right)}dx$$

$$= \int \frac{1}{1-x}dx + \int \frac{x}{1+x^2}\, dx + \int \frac{1}{1+x^2} \quad [1]$$

$$= \frac{\log|1-x|}{-1} + \frac{1}{2}\int \frac{dp}{p} + \tan^{-1}(x)$$

$$= -\log|1-x| + \frac{1}{2}\log|p| + \tan^{-1} x + c$$

$$= -\log|1-x| + \frac{1}{2}\log\left|x^2 + 1\right| + \tan^{-1} x + c \quad [1]$$

20. Given: $I=\int \frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}dx$

using: $\left[\int \frac{dx}{x^2-a^2}=\frac{1}{2a}\ \log\left|\frac{x-a}{x+a}\right|+C\right.$

$=\int \frac{dp}{(p+1)(p+3)}$ [1]

$=\int \frac{dp}{p^2+4p+3}$

$=\int \frac{dp}{p^2+4p+3}$

$=\int \frac{dp}{p^2+4p+(2)^2+3-(2)^2}$ [1]

$=\int \frac{dp}{(p+2)^2-(1)^2}$

$=\frac{1}{2\times 1}\log\left|\frac{P+2-1}{P+2+1}\right|+C$ [1]

$=\frac{1}{2}\ \log\left|\frac{p+1}{p+3}\right|+C$

$=\frac{1}{2}\ \log\left(\frac{x^2+1}{x^2+3}\right)+C$ [1]

21. Let $\mathrm{I}=\int e^{2x}\sin x\,dx$

Integrating by parts

$I=\sin x\int e^{2x}dx\int\left\{\frac{d}{dx}(\sin x)\int e^{2x}dx\right\}dx$ [1]

$I=\sin x\frac{e^{2x}}{2}-\int\cos x\left(\frac{e^{2x}}{2}\right)dx$

$I=\sin x\frac{e^{2x}}{2}-\frac{1}{2}\int\cos xe^{2x}.dx$ [1]

Again integrating by parts, we obtain

$I=\sin x\frac{e^{2x}}{2}-\frac{1}{2}\left[\cos x\int e^{2x}.dx-\int\left\{\begin{matrix}\frac{d}{dx}(\cos x)\\ \int e^{2x}dx\end{matrix}\right\}dx\right]$

[1]

$=\sin x\frac{e^{2x}}{2}-\frac{1}{2}\left[\cos x.\left(\frac{e^{2x}}{2}\right)+\frac{1}{2}\int\sin x.e^{2x}dx\right]$

$I=\sin x\frac{e^{2x}}{2}-\frac{1}{2}\left[\cos x.\left(\frac{e^{2x}}{2}\right)+\frac{1}{2}I\right]$

$I=\sin x\frac{e^{2x}}{2}-\frac{1}{2}\cos x\left(\frac{e^{2x}}{2}\right)-\frac{1}{4}I$

$\frac{5}{4}I=\sin x\frac{e^{2x}}{2}-\frac{1}{2}\cos x.\left(\frac{e^{2x}}{2}\right)$

$I=\frac{4}{5}\left[\sin x\frac{e^{2x}}{2}-\frac{1}{2}\cos x.\left(\frac{e^{2x}}{2}\right)\right]$

$I=\frac{e^{2x}}{5}[2\sin x-\cos x]+C$ [1]

22. $\int\frac{x+3}{\sqrt{x^2-2x-5}}dx$

Let $x+3=A\frac{d}{dx}\left(x^2-2x-5\right)+B$

$x+3=A(2x-2)+B$

$x+3=2Ax-2A+B$ [1]

On equating coefficient

$x=2Ax$

$A=\frac{1}{2}$

$3=-2A+B$

$3=-2\left(\frac{1}{2}\right)+B$

B = 4 [1]

Now,

$\int\frac{x+3}{\sqrt{x^2-2x-5}}dx$

$\int\frac{\frac{1}{2}(2x-2)+4}{\sqrt{x^2-2x-5}}dx$

$\Rightarrow\frac{1}{2}\int\frac{(2x-2)}{\sqrt{x^2-2x-5}}dx+4\int\frac{1}{\sqrt{x^2-2x-5}}dx$

$A+B \quad ...(1)$

$A=\frac{1}{2}\int\frac{(2x-2)}{\sqrt{x^2-2x-5}}dx$

Let $x^2-2x-5=t$

$(2x-2)\,dx=dt$

Now, $\frac{1}{2}\int\frac{(2x-2)}{\sqrt{x^2-2x-5}}dx=\frac{1}{2}\int\frac{1}{\sqrt{t}}dt$

$=\frac{1}{2}2\sqrt{t}$

$=\frac{1}{2}\left(2\sqrt{(x^2-2x-5}\right)$ [1]

$B=4\int\frac{1}{\sqrt{x^2-2x-5}}dx$

$=4\int\frac{1}{\sqrt{(x^2-2x-1)-6}}dx$

$=4\int\frac{1}{\sqrt{(x-1)^2-(\sqrt{6})^2}}dx$

$=4\log\left|x-1+\sqrt{x^2-2x-5}\right|$

$=4\log\left|x-1+\sqrt{x^2-2x-5}\right|$

Put value of A & B in equation (1)

$A+B=\frac{1}{2}\left(2\sqrt{(x^2-2x-5)}\right)+C$ [1]

23. $\int\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}dx$

$=\int\frac{\left(2\cos^2 x-1\right)-\left(2\cos^2\alpha-1\right)}{\cos x-\cos\alpha}dx$ [1]

$=\int\frac{2(\cos x-\cos\alpha)(\cos x-\cos\alpha)}{(\cos x-\cos\alpha)}dx$ [1]

$=2\int(\cos x+\cos\alpha\, dx)$ [1]

$=2(\sin x+x\cos\alpha)+c$ [1]

24. $I=\int\frac{x+2}{\sqrt{x^2+2x+3}}dx$

$I=\int\left\{\frac{(x+1)+1}{\sqrt{x^2+2x+3}}\right\}dx$ [1]

$I=\int\left\{\frac{(x+1)}{\sqrt{x^2+2x+3}}\right\}dx+\int\left\{\frac{1}{\sqrt{x^2+2x+3}}\right\}dx$ [1]

$I=I_1+I_2$

In I_1 let $x^2+2x+3=t^2$

$\therefore(2x+2)\,dx=2t\,dt$

$\Rightarrow(x+1)\,dx=t\,dt$

$\therefore I_1=\int\frac{t.dt}{t}=t$ [1]

$I_1=\sqrt{x^2+2x+3}$

Now $I=I_1+I_2$

$\therefore I=\sqrt{x^2+2x+3}+\left|\log(x+1)+\sqrt{x^2+2x+3}\right|+c$

[1]

25. $I=\int\frac{dx}{x\left(x^5+3\right)}$

$I=\int\frac{x^4dx}{x\left(x^5+3\right)}$ [1]

Let $x^5=t$

$5x^4=\frac{dt}{dx}$

$\Rightarrow x^4dx=\frac{dt}{5}$ [1]

$I=\frac{1}{5}\int\frac{dt}{t(t+3)}$

$I=\frac{1}{5}.\frac{1}{3}\int\left(\frac{1}{t}-\frac{1}{t+3}\right)dt$

$I=\frac{1}{15}\{\log t-\log(t+3)\}+c$ [1]

$I=\frac{1}{15}\log\left(\frac{t}{t+3}\right)+c$

$I=\frac{1}{15}\log\left(\frac{x^5}{x^5+3}\right)+c$ [1]

26. Given : $\int \frac{x+2}{\sqrt{x^2+5x+6}}dx$

Put, $x+2=\lambda\frac{d}{dx}(x^2+5x+6)+\mu$ [1]

$x+2=2x\lambda+5\lambda+\mu$

Comparing coefficients of x both sides

$1 = 2\lambda$

$\Rightarrow \lambda = 1/2$

Comparing constant terms both sides,

$2 = 5\lambda+\mu$

Or, $= 2-\frac{5}{2}=-\frac{1}{2}$

$\therefore \int \frac{x+2}{\sqrt{x^2+5x+6}}dx=\frac{1}{2}\int\frac{2x+5}{\sqrt{x^2+5x+6}}dx$

I= I$_1$

$-\frac{1}{2}\int\frac{1}{\sqrt{x^2+5x+6}}dx$

- I$_2$

$\therefore I=I_1-I_2$ ...(1)

$I_1=\frac{1}{2}\int\frac{2x+5}{\sqrt{x^2+5x+6}}dx$ [1]

Put $x^2+5x+6=t$

$\therefore (2x+5)\,dx=dt$

$=\frac{1}{2}\int\frac{dt}{\sqrt{t}}=\frac{1}{2}\left(\frac{t^{-\frac{1}{2}}}{-\frac{1}{2}+1}\right)+C=t^{\frac{1}{2}}+C=\sqrt{x^2+5x+6}+C$

$I_2=\frac{1}{2}\int\frac{1}{\sqrt{x^2+5x+6}}dx$ [1]

$=\frac{1}{2}\int\frac{dx}{\sqrt{x^2+5x+\frac{25}{4}-\frac{25}{4}+6}}=\frac{1}{2}\int\frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$

$=\frac{1}{2}\log\left[\left(x+\frac{5}{2}\right)+\sqrt{\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right]+C$

$=\frac{1}{2}\log\left[\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right]+C$

Substituting the values of I_1 and I_2 in (1) we get,

$I=\sqrt{x^2+5x+6}+\frac{1}{2}.\log\left|\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right|+C$

[1]

27. $\int\frac{x^3-1}{x(x^2+1)}dx=\int\left(1-\frac{x+1}{x\left(x^2+1\right)}\right)dx$

$=x-\int\frac{x+1}{x\left(x^2+1\right)}dx$ [1]

$=x-I_1$

Let $\frac{x+1}{x\left(x^2+1\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}=\frac{1}{x}+\frac{1-x}{x^2+1}$ [1]

$\therefore I_1=\int\frac{1}{x}+\frac{1-x}{x^2+1}dx=\log x-\frac{1}{2}\log\left|x^2+1\right|+\tan^{-1}x$

[1]

$\therefore I=x-\log|x|+\frac{1}{2}\log\left|x^2+1\right|-\tan^{-1}x+c$ [1]

28. $\int(x+3)\sqrt{3-4x-x^2}\,dx$

$\Rightarrow x+3=p\frac{d}{dx}\left(3-4x-x^2\right)+q$ [1]

$\Rightarrow x+3=p(-4-2x)+q$

So, $-2p=1\Rightarrow p=-\frac{1}{2}$

$-4p+q=3$

$\Rightarrow q=1$

So given integral

$\int\left\{-\frac{1}{2}(-4-2x)+1\right\}\sqrt{3-4x-x^2}\,dx$

$\Rightarrow\int-\frac{1}{2}(-4-2x)\sqrt{3-4x-x^2}\,dx+\int\sqrt{3-4x-x^2}\,dx$

[1]

Put $3-4x-x^2=t$

$-4-2x=\frac{dt}{dx}$

$\Rightarrow (-4-2x)\,dx = dt$

$\Rightarrow -\frac{1}{2}\int\sqrt{t}dt + \int\sqrt{7-(x+2)^2}\,dx$ [1]

$\Rightarrow -\frac{1}{2}.\frac{2}{3}t^{3/2} + \frac{x+2}{2}\sqrt{7-(x+2)^2} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + c$

$\Rightarrow -\frac{1}{3}(3-4x-x^2)^{\frac{3}{2}} + \left(\frac{x+2}{2}\right)\sqrt{3-4x-x^2}$

$+\frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + c$ [1]

29. Given, $\int\frac{(2x-5)\,e^{2x}}{(2x-3)^3}\,dx$

put $2x = t$ [1]

$= \frac{1}{2}\int\frac{(t-5)\,e^t}{(t-3)^3}\,dt$

$= \frac{1}{2}\int e^t\left(\frac{t-3}{(t-3)^3} - \frac{2}{(t-3)^3}\right)dt$ [1]

$= \frac{1}{2}\int e^t\left(\frac{1}{(t-3)^2} - \frac{2}{(t-3)^3}\right)dt$

$\because \int e^x\left(f(x)+f'(x)\right)dx = e^x f(x) + c$ [1]

$= \frac{1}{2}e^t\frac{1}{(t-3)^2} + c$

$= \frac{1}{2}e^{2x}\frac{1}{(2x-3)^2} + c$ [1]

30. $\int\left(\frac{Ax+B}{(x^2+1)} + \frac{C}{(x+2)}\right)dx$

$\Rightarrow x^2+x+1 = (Ax+B)(x+2) + C(x^2+1)$

$\Rightarrow x^2+x+1 = Ax^2+2Ax+Bx+2B+Cx^2+C$ [1]

$A+C=1$ ...(1)

$2A+B=1$ ...(2)

$2B+C=1$ ...(3)

By (1) , (2) and (3) we get

$C = \frac{3}{5}$

$\therefore B = \frac{6}{5} - 1 \Rightarrow B = \frac{1}{5}$

$\therefore A = 1 - \frac{3}{5} = \frac{2}{5}$ [1]

$= \int\left(\frac{1}{5}.\frac{(2x+1)}{(x^2+1)} + \frac{3}{5}\frac{1}{(x+2)}\right)dx$

$= \frac{1}{5}\int\frac{2x}{x^2+1}\,dx + \frac{1}{5}\int\frac{1}{x^2+1}\,dx + \frac{3}{5}\int\frac{1}{x+2}\,dx$ [1]

$= \frac{1}{5}\log\left(x^2+1\right) + \frac{1}{5}\tan^{-1}x + \frac{3}{5}\log|x+2| + c$ [1]

31. $\int\frac{\cos\theta}{(4+\sin^2\theta)(5-4+4\sin^2\theta}\,d\theta$

$= \int\frac{\cos\theta}{\left(4+\sin^2\theta\right)\left(1+4\sin^2\theta\right)}\,d\theta$ ...(1)

Let

$\sin\theta = t$

$\cos\theta d\theta = dt$ [1]

So, $I = \int\frac{dt}{\left(4+t^2\right)\left(1+4t^2\right)}$ ...(2)

Let $t^2 = y$ and thus applying partial fraction we get.

$\frac{1}{(4+y)(1+4y)} = \frac{A}{4+y} + \frac{B}{1+4y}$ [1]

$1 = A(1+4y) + B(4+y)$

$A = -\frac{1}{15},\ B = \frac{4}{15}$

Putting the values in (2) we get,

$$I = \int \left| \frac{-\frac{1}{15}}{4+t^2} + \frac{\frac{4}{15}}{1+4t^2} \right| dt$$

$$\therefore I = \int -\frac{1}{15}\left[\frac{1}{4+t^2}\right]dt + \int \frac{4}{15}\left[\frac{1}{1+4t^2}\right]dt$$

$$\therefore I = -\frac{1}{15}\int \frac{dt}{4+t^2} + \frac{4}{15}\int \frac{dt}{1+4t^2}$$ [1]

$$\therefore I = -\frac{1}{15}\int \frac{dt}{4+t^2} + \frac{1}{15}\int \frac{dt}{\frac{1}{4}+t^2}$$

$$\therefore I = -\frac{1}{15}\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right] + \frac{1}{15}\left[2\tan^{-1}2t\right] + C$$

$$\therefore I = -\frac{1}{30}\tan^{-1}\left(\frac{\sin\theta}{2}\right) + \frac{2}{15}\tan^{-1}(2\sin\theta) + C$$

[1]

32. $\int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$

Let $t = \sin x$

$$\frac{dt}{dx} = \cos x$$

$dt = \cos x dx$ [1]

$$\Rightarrow \int \frac{2dt}{(1-t)(1+t^2)}$$

Applying partial fraction we get,

Let $\frac{2}{(1-t)(1+t^2)} = \frac{A}{1-t} + \frac{Bt+C}{1+t^2}$ ...(1)

$$\frac{2}{(1-t)(1+t^2)} = \frac{A(1+t^2)+(Bt+C)(1-t)}{(1-t)(1+t^2)}$$

$$2 = A(1+t^2)+(Bt+C)(1-t)$$

$2 = A + At^2 + Bt + C - Bt^2 - Ct$ [1]

Comparing both sides we get,

Coefficient of t^2

$0 = A - B$

$A = B$ ...(2)

Coefficient of t

$0 = B - C$

$B = C$ ...(3)

Constant term

$2 = A + C$

From (2) and (3)

$2 = B + B$

$B = 1$

$\Rightarrow A = B = C = 1$

Substituting the values of A,B,C in equation (1) we get

$$\frac{2}{(1-t)(1+t^2)} = \frac{1}{1-t} + \frac{t+1}{1+t^2}$$

$$\int \frac{2}{(1-t)(1+t^2)}dt = \int\left(\frac{1}{1-t} + \frac{t+1}{1+t^2}\right)dt$$

$= -\log|1-t| + \frac{1}{2}\int \frac{2t}{t^2+1}dt + \int \frac{dt}{t^2+1}$ [1]

Using $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C$ and

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$= -\log|1-t| + \frac{1}{2}\log\left|t^2+1\right| + \tan^{-1}\frac{t}{1} + C$$

Substituting the value $t = \sin x$

$$= -\log|1-\sin(x)| + \frac{1}{2}\log\left|\sin^2 x + 1\right| + \tan^{-1}(\sin x) + C$$

[1]

33. $\int \frac{dx}{\sin x + \sin 2x}$

$\int \frac{dx}{\sin x + 2\sin x \cos x}$

$\int \frac{dx}{\sin x(1+2\cos x)}$

Multiplying numerator and denominator by sin x.

$= \int \frac{\sin x\, dx}{\sin^2 x(1+2\cos x)}$

$= \int \frac{\sin x\, dx}{(1-\cos^2 x)(1+2\cos x)}$ [1]

Let $p = \cos x, -dp = \sin x\, dx$

$= -\int \frac{dp}{(1-p^2)(1+2p)}$

$= \int \frac{dp}{(p^2-1)(1+2p)}$

$= \int \frac{dp}{(p+1)(p-1)(1+2p)}$

Let

$\frac{dp}{(p-1)(p+1)(1+2p)} = \frac{A}{(p-1)} + \frac{B}{(p+1)} + \frac{C}{(2p+1)}$ ....(i) [1]

$1 = A(p+1)(2p+1) + B(p-1)(2p+1) + C(p-1)(p+1)$

$1 = A(2p^2+3p+1) + B(2p^2-p-1) + C(p^2-1)$

Comparing the coefficients of p^2, p and constant term.

we get $A = \frac{1}{6}$

$B = \frac{1}{2}$ and

$C = \frac{-4}{3}$

Putting the value of A,B and C in (i), we get

$\int \frac{dp}{(p+1)(p-1)(1+2p)} = \int \left(\frac{\frac{1}{6}}{p-1} + \frac{\frac{1}{2}}{p+1} + \frac{\frac{-4}{3}}{2p+1} \right) dp$

$= \frac{1}{6}\log|p-1| + \frac{1}{2}\log|p+1| - \frac{4}{2\times 3}\log|2p+1| + c$

$= \frac{1}{6}\log|\cos x - 1| + \frac{1}{2}\log|\cos x + 1| - \frac{2}{3}\log |2\cos x + 1| + c$ [1]

34. Given $\int \frac{1}{\cos^4 x + \sin^4 x} dx$

I = $\int \frac{\frac{1}{\cos^4 x} dx}{1+\tan^4 x}$

= $\int \frac{\sec^2 x \sec^2 x dx}{1+\tan^4 x}$ [1]

= $\int \frac{(1+\tan^2 x)\sec^2 x dx}{1+\tan^4 x}$

Put tan $x = t$

$\therefore$ $\sec^2 x dx = dt$

$\therefore$ $I = \int \frac{(1+t^2)\, dt}{1+t^4}$ [1]

Dividing numerator and denominator by (t^2)

$= \int \frac{\left(\frac{1}{t^2}+1\right) dt}{\frac{1}{t^2}+t^2}$

$= \int \frac{\left(1+\frac{1}{t^2}\right) dt}{\left(t-\frac{1}{t}\right)^2+2}$ [1]

put $t-\frac{1}{t}=z$ [1]

$\therefore \left(1+\frac{1}{t^2}\right)dt=dz$

$\therefore \; I=\int\frac{dz}{z^2+2}=\frac{1}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}}+C$

$=\frac{1}{\sqrt{2}}\tan^{-1}\frac{\left(t-\frac{1}{t}\right)}{\sqrt{2}}+C$ [1]

$=\frac{1}{\sqrt{2}}\tan^{-1}\frac{\left(\tan x-\frac{1}{\tan x}\right)}{\sqrt{2}}+C$

$=\frac{1}{\sqrt{2}}\tan^{-1}\frac{(\tan x-\cot x)}{\sqrt{2}}+C$ [1]

35. $I=\int(\sqrt{\cot x}+\sqrt{\tan x})dx$

$=\int\frac{\cos x+\sin x}{(\sqrt{\sin x\cos x})}dx$ [1]

$=\sqrt{2}\int\frac{\cos x+\sin x}{(\sqrt{1-(1-2\sin x\cos x)}}dx$ [1]

$=\sqrt{2}\int\frac{\cos x+\sin x}{(\sqrt{1-(\sin x-\cos x)^2}}dx$ [1]

Put $\sin x-\cos x=t$ then

$\therefore \; (\cos x+\sin x)\,dx=dt$ [1]

$\therefore I=\sqrt{2}\int\frac{dt}{\sqrt{1-t^2}}$

$=\sqrt{2}\sin^{-1}t+c$ [1]

$=\sqrt{2}\sin^{-1}(\sin x-\cos x)+c$ [1]

[TOPIC 2] Properties of a Definite Integrals and Limit of a Sum

Summary

- **Definite Integrals:**

  A definite integral is denoted by $\int_a^b f(x)dx$, where a is called the lower limit of the integral and b is called the upper limit of the integral.

- **Definite integral as the limit of a sum:**

  The definite integral $\int_a^b f(x)dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis.This can be mathematically defined as following:

  $$\int_a^b f(x)\,dx = (b-a)\lim_{h\to 0}\frac{1}{n}\left[f(a)+f(a+h) + \ldots + f(a+(n-1)h)\right]$$

  Where, $h = \frac{b-a}{n} \to 0$ as $n \to \infty$ $\int_a^b f(x)\,dx$ is defined as the area function where the area of the region is bounded by the curve $y = f(x)$, $a \le x \le b$, the $x - axis$ and the ordinates $x = a$ and $x = b$. Let x be a given point in $[a, b]$. Then $\int_a^x f(x)\,dx$ represents the **Area function $A(x)$**.

- **First fundamental theorem of integral calculus:**

  Let f be a continuous function on the closed interval $[a, b]$ and let $A(x) = \int_a^b f(x)\,dx$ for all $x \ge a$ be the area function. Then $A'(x) = f(x)$, for all $x \in [a, b]$.

- **Second fundamental theorem of integral calculus:**

  Let f be continuous function defined on the closed interval $[a, b]$ and F be an anti derivative of f. Then

  $$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a).$$

- **Properties of Definite Integrals** are as follows:

  ➢ $P_0 : \int_a^b f(x)dx = \int_a^b f(t)\,dt$

  ➢ $P_1 = \int_a^b f(x)\,dx = -\int_b^a f(x)dx.$

  In particular $\int_a^a f(x)\,dx = 0$

  ➢ $P_2 : \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

  ➢ $P_3 : \int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

  ➢ $P_4 : \int_0^a f(x)dx = \int_0^a f(a-x)dx$

  ➢ $P_5 : \int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)\,dx$

  ➢ $P_6 : \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx,$ if $(2a - x) = f(x)$ and 0, if $(2a - x) = -f(x)$.

  ➢ $P_7 : \int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, \text{ if f is an even function, i.e., } f(-x) = f(x)(2a-x) = f(x) \\ 0, \text{ if f is an odd function, i.e., if } f(-x) = -f(x) \end{cases}$

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 2

1 Mark Questions

1. Evaluate $\int_2^3 3^x\,dx$

[DELHI 2017]

2. Evaluate: $\int_0^3 \frac{dx}{9+x^2}$

[DELHI 2014]

3. Evaluate: $\int_2^3 \frac{1}{x}dx$

[ALL INDIA 2012]

4. Evaluate $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$

[ALL INDIA 2011]

5. If $f(x)=\int_0^x t\sin t\,dt$, then write the value of $f'(x)$.

[ALL INDIA 2014]

6. Evaluate: $\int_2^4 \frac{x}{x^2+1}dx$

[ALL INDIA 2014]

4 Marks Questions

7. Evaluate: $\int_0^4 \left(|x|+|x-2|+|x-4|\right)dx$.

[DELHI 2013]

8. Evaluate: $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$

[DELHI 2017]

9. Evaluate: $\int_0^{\frac{3}{2}} |x\sin \pi x|\,dx$

[DELHI 2017]

10. Evaluate $\int_0^\pi e^{2x}.\sin\left(\frac{\pi}{4}+x\right)dx$

[DELHI 2016]

11. Evaluate $\int_{-1}^2 \left|x^3-x\right|dx$

[DELHI 2016]

12. Evaluate: $\int_0^{2\pi} \frac{1}{1+e^{\sin x}}dx$

[ALL INDIA 2013]

13. Evaluate: $\int_0^\pi \frac{4x\sin x}{\left(1+\cos^2 x\right)}dx$

[ALL INDIA 2014]

14. Evaluate: $\int_0^{\frac{3}{2}} |x.\cos(\pi x)|\,dx$

[ALL INDIA 2015]

15. Evaluate: $\int_{-2}^2 \frac{x^2}{1+5^x}dx$

[ALL INDIA 2016]

16. $\int_0^\pi \frac{x\tan x}{\sec x+\tan x}dx$

[ALL INDIA 2017]

17. Evaluate: $\int_{-\pi}^{\pi} (\cos ax-\sin bx)^2\,dx$.

[DELHI 2015]

6 Marks Questions

18. Evaluate $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$

[DELHI 2014]

19. Evaluate $\int_1^3 \left(e^{2-3x}+x^2+1\right)dx$ as a limit of *a* sum.

[DELHI 2015]

20. Evaluate: $\int_o^{\pi/2} 2\sin x\cos x\tan^{-1}(\sin x)\,dx$

[DELHI 2011]

21. Evaluate: $\int_0^{\frac{\pi}{2}} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}dx$

[DELHI 2011]

22. Evaluate $\int_1^4 \left(x^2-x\right)dx$ as a limit of sum.

[ALL INDIA 2011]

23. Evaluate: $\int_0^{\pi} \log\left(1+\cos x\right)dx$

[DELHI 2011]

24. Prove that: $\int_0^{\frac{\pi}{4}} \left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx = \sqrt{2}.\frac{\pi}{2}$

[ALL INDIA 2012]

25. Evaluate: $\int_0^{\frac{\pi}{2}} \frac{2^{\text{sinx}}}{2^{sinx}+2^{cosx}}dx$

[ALL INDIA 2015]

26. Evaluate $\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{16+9\sin 2x}dx$

[DELHI 2018]

27. Evaluate $\int_1^3 \left(x^2+3x+e^x\right)dx$ as the limit of the sum.

[DELHI 2018]

Solutions

1. $I = \int_2^3 3^x\,dx$

$= \frac{1}{\log(3)}\left[3^x\right]_2^3$

$= \frac{1}{\log(3)}\left[3^3-3^2\right]$

$= \frac{1}{\log(3)}\left[27-9\right]$

$= \frac{18}{\log(3)}$ [½]

Using $1 = \log e$

$= \frac{18\log e}{\log(3)} = 18\log_3 e$ [½]

2. $\int_0^3 \frac{dx}{9+x^2}$

$I = \int_0^3 \frac{dx}{3^2+x^2}$

$= \frac{1}{3}\left[\tan^{-1}\left(\frac{x}{3}\right)\right]_0^3$ $\left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$

[½]

$= \frac{1}{3}\left[\tan^{-1}\left(\frac{3}{3}\right)-\tan^{-1}\left(\frac{0}{3}\right)\right]$

$= \frac{1}{3}\left[\tan^{-1}(1)-\tan^{-1}(0)\right]$

$\frac{1}{3}\left[\frac{\pi}{4}-0\right] = \frac{\pi}{12}\left[\tan^{-1}(1) = \frac{\pi}{4}, \tan^{-1}(0) = 0\right]$ [½]

3. Given : $\int_2^3 \frac{1}{x}dx = \left[\log|x|\right]_2^3$

$= \log 3 - \log 2 = \log\frac{3}{2}$ [1]

4. $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$

Let $I = \int_1^{\sqrt{3}} \frac{dx}{1+x^2}$

$= \left[\tan^{-1}x\right]_1^{\sqrt{3}}$ [½]

$= \tan^{-1}(\sqrt{3})-\tan^{-1}(1)$

$= \frac{\pi}{3}-\frac{\pi}{4}$

$= \frac{\pi}{12}$ [½]

5. $f(x) = \int_0^x t\sin t\,dt$

$\Rightarrow f'(x) = 1.x\sin x - 0$

$= x\sin x$ [1]

6. $I = \int_2^4 \frac{x}{x^2+1}dx$

Put $x^2+1 = t$ [½]

$2x = \frac{dt}{dx}$

$\Rightarrow 2xdx = dt$

$xdx = \frac{1}{2}dt$

at $x = -2,\ t = 5$

at $x = 4,\ t = 17$

$\therefore I = \int_5^{17} \frac{1/2}{t}dt$

$= \frac{1}{2}\left[\log|t|\right]_5^{17}$

$= \frac{1}{2}\left[\log 17 - \log 5\right]$

$= \frac{1}{2}\log\frac{17}{5}$ [½]

7. Given: $\left| x = \begin{cases} -x, x < 0 \\ x, \quad x \le 0 \end{cases} \right|$

$|x-2| = \begin{cases} -(x-2), x < 2 \\ (x-2), x \ge 2 \end{cases} = \begin{cases} (2-x), x < 2 \\ (x-2), x \ge 2 \end{cases}$

$|x-4| = \begin{cases} -(x-4), x < 4 \\ (x-4), x \ge 4 \end{cases} = \begin{cases} (4-x), x < 4 \\ (x-4), x \ge 4 \end{cases}$ [1]

$\int(|x| + |x-2| + |x-4|)dx$

$\left[\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\right]$ *where* $a < c < b$

[1]

$= \int_0^4 xdx + \int_0^2 (2-x)\,dx + \int_2^4 (x-2)\,dx + \int_0^4 (4-x)\,dx$

$= \frac{1}{2}\left(x^2\right)_0^4 + \left(2x - \frac{x^2}{2}\right)_0^2 + \left(\frac{x^2}{2} - 2x\right)_2^4 + \left(4x - \frac{x^2}{2}\right)_0^4$

[1]

$= \frac{1}{2}(16-0) + \left(4 - \frac{4}{2}\right) - (0) + \left(\frac{16}{2} - 8\right) - \left(\frac{4}{2} - 4\right)$

$+ \left(16 - \frac{16}{2}\right) - (0)$

= 8 + 2 + 0 + 2 + 8 = 20 [1]

8. Given $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$

Let $I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$

Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ we get

$I = \int_0^\pi \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx$ [1]

$I = \int_0^\pi \frac{(\pi - x)\sin x}{1+\cos^2 x}dx$

$I = \int_0^\pi \left(\frac{\pi\sin x}{1+\cos^2 x} - \frac{x\sin x}{1+\cos^2 x}\right)dx$

$I = \int_0^\pi \left(\frac{\pi\sin x}{1+\cos^2 x}\right)dx - \int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$ [1]

From (1) we get

$I = \int_0^\pi \left(\frac{\pi\sin x}{1+\cos^2 x}\right)dx - I$

$2I = \int_0^\pi \left(\frac{\pi\sin x}{1+\cos^2 x}\right)dx$

Let $\cos x = t$, when $x = \pi$ then $t^2 - 1$, when $x = 0$ then $t = 1$ [1]

$-\sin x = \frac{dt}{dx}$

$\sin xdx = -dt$

$2I = \int_1^{-1} \frac{-\pi}{1+t^2}dt$

$$2I = -\pi\left[\tan^{-1} t\right]_1^{-1}$$

$$2I = -\pi\left[\tan^{-1}(-1) - \tan^{-1}(1)\right]$$

$$2I = -\pi\left[\frac{-\pi}{2}\right] = \frac{\pi^2}{2}$$

$$I = \frac{\pi^2}{4}$$ [1]

9. Given $\int_0^{\frac{3}{2}} |x\sin \pi x|\, dx$

$|x\sin \pi x| = x\sin \pi x,$ for $0 \le x \le 1$

$|x\sin \pi x| = -x\sin \pi x,$ for $1 \le x \le \frac{3}{2}$ [1]

Using $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$$\int_0^{\frac{3}{2}} |x\sin \pi x|\,dx = \int_0^1 x\sin \pi x dx + \int_1^{\frac{3}{2}} (-x\sin \pi x)\,dx$$ [1]

Using $I\int II\,dx - \int\left[\frac{dI}{dx}\int II\,dx\right]dx$

$$= \left[x\int \sin \pi x\,dx - \int\left(\frac{d}{dx}(x)\int \sin \pi x dx\right)\right]_0^1$$

$$- \left[x\int \sin \pi x\,dx - \int\left(\frac{d}{dx}(x)\int \sin \pi x dx\right)\right]_1^{\frac{3}{2}}$$

$$= \left[\frac{-x\cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2}\right]_0^1 - \left[\frac{-x\cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2}\right]_1^{\frac{3}{2}}$$

$$= \left[\frac{-\cos \pi}{\pi} + \frac{\sin \pi}{\pi^2}\right] - \left[0 + \frac{\sin 0}{\pi^2}\right] -$$

$$\left[\left(\frac{-\frac{3}{2}\cos\left(\pi\frac{3}{2}\right)}{\pi} + \frac{\sin\left(\pi\frac{3}{2}\right)}{\pi^2}\right) - \left(\frac{-\cos \pi}{\pi} + \frac{\sin \pi}{\pi^2}\right)\right]$$

[1]

$$= \left[\frac{-1(-1)}{\pi} + 0\right] - [0+0] - \left[\left(0 + \frac{-1}{\pi^2}\right) - \left(\frac{-1(-1)}{\pi} + 0\right)\right]$$

$$= \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi}$$

$$= \frac{2}{\pi} + \frac{1}{\pi^2}$$ [1]

10. Let I = $\int e^{2x}.\sin\left(\frac{\pi}{4} + x\right)dx$ (1)

Integrating by parts taking e^{2x} as IInd function,

$$= \sin\left(\frac{\pi}{4} + x\right).\frac{e^{2x}}{2} - \int \cos\left(\frac{\pi}{4} + x\right)\frac{e^{2x}}{2}\,dx$$

$$= \sin\left(\frac{\pi}{4} + x\right).\frac{e^{2x}}{2} - \frac{1}{2}\int \cos\left(\frac{\pi}{4} + x\right)e^{2x}dx$$ [1]

Again integrating by parts taking e^{2x} as IInd function,

$$= \sin\left(\frac{\pi}{4} + x\right).\frac{e^{2x}}{2} - \frac{1}{2}\left[\cos\left(\frac{\pi}{4} + x\right)\frac{e^{2x}}{2} - \int -\sin\left(\frac{\pi}{4} + x\right)\frac{e^{2x}}{2}\,dx\right]$$

$$= \sin\left(\frac{\pi}{4} + x\right).\frac{e^{2x}}{2} - \frac{1}{4}\cos\left(\frac{\pi}{4} + x\right)e^{2x}$$

$$- \frac{1}{4}\int \sin\left(\frac{\pi}{4} + x\right)e^{2x}dx$$

$$I = \frac{e^{2x}}{4}\left[2\sin\left(\frac{\pi}{4} + x\right) - \cos\left(\frac{\pi}{4} + x\right)\right] - \frac{1}{4}I$$ [1]

using (1)

$$I + \frac{1}{4}I = \frac{e^{2x}}{4}\left[2\sin\left(\frac{\pi}{4} + x\right) - \cos\left(\frac{\pi}{4} + x\right)\right]$$

$$\frac{5I}{4} = \frac{e^{2x}}{4}\left[2\sin\left(\frac{\pi}{4} + x\right) - \cos\left(\frac{\pi}{4} + x\right)\right]$$

$$I = \frac{e^{2x}}{5}\left[2\sin\left(\frac{\pi}{4} + x\right) - \cos\left(\frac{\pi}{4} + x\right)\right]$$

$$\int_0^{\pi} e^{2x}\sin\left(\frac{\pi}{4}+x\right) = \frac{1}{5}\left[e^{2x}\begin{pmatrix}2\sin\left(\frac{\pi}{4}+x\right)\\ -\cos\left(\frac{\pi}{4}+x\right)\end{pmatrix}\right]_0^{\pi}$$

$$= \frac{1}{5}\left[e^{2\pi}\left(2\sin\frac{5\pi}{4}-\cos\frac{5\pi}{4}\right)-e^{0}\left(2\sin\frac{\pi}{4}-\cos\frac{\pi}{4}\right)\right]$$ [1]

$$\frac{1}{5}\left[e^{2\pi}\left(\frac{-2}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)-1\left(2\times\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{1}{5}\left[e^{2\pi}\left(\frac{-1}{\sqrt{2}}\right)-\left(\frac{1}{\sqrt{2}}\right)\right] = \frac{-1}{5\sqrt{2}}\left(e^{2\pi}+1\right)$$ [1]

11. Let $f(x) = x^3 - x = x(x^2-1)$

$= x(x-1)(x+1)$

$$|x^3 - x| = \begin{cases} +(x^3-x), [-1,0] \\ -(x^3-x), [0,1] \\ +(x^3-x), [1,2] \end{cases}$$ [1]

$$\therefore \int_{-1}^{2} |x^3-x|\,dx$$

$$= \int_{-1}^{0}(x^3-x)\,dx + \int_0^1 -(x^3-x)\,dx + \int_1^2 (x^3-x)\,dx$$

$$= \left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1 + \left[\frac{x^4}{4}-\frac{x^2}{2}\right]_1^2$$ [2]

$$= (0)-\left(\frac{1}{4}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)-0+(4-2)-\left(\frac{1}{4}-\frac{1}{2}\right)$$

$$= -\frac{1}{4}+\frac{1}{2}+\frac{1}{2}-\frac{1}{4}+2-\frac{1}{4}+\frac{1}{2} = \frac{3}{2}-\frac{3}{4}+2$$

$$= \frac{11}{4}$$ [1]

12. $I = \int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx \qquad ...(1)$

$$I = \int_0^{2\pi} \frac{1}{1+e^{\sin(2\pi - x)}}\,dx$$ [1]

$$I = \int_0^{2\pi} \frac{1}{1+e^{-\sin x}}\,dx$$

$$I = \int_0^{2\pi} \frac{e^{\sin x}}{1+e^{\sin x}}\,dx \qquad ...(2)$$ [1]

Adding (1) and (2) we get

$$\Rightarrow 2I = \int_0^{2\pi}\left(\frac{1+e^{\sin x}}{1+e^{\sin x}}\right)dx$$ [1]

$$\Rightarrow 2I = [x]_0^{2\pi}$$

$$\Rightarrow 2I = 2\pi$$

$$I = \pi$$ [1]

13. $\mathrm{I} = \int_0^{\pi} \frac{4x\sin x}{(1+\cos^2 x)}\,dx$

Applying $\int f(a-x) = \int f(x)\}$

$$I = \int_0^{\pi} \frac{4(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\,dx$$ [1]

$$I = \int_0^{\pi} \frac{4\pi\sin x}{(1+\cos^2 x)}\,dx - \int_0^{\pi}\frac{4x\sin x}{(1+\cos^2 x)}\,dx$$

$$\mathrm{I} = \int_0^{\pi}\frac{4\pi\sin x}{(1+\cos^2 x)}\,dx - \mathrm{I}$$

$$2\mathrm{I} = 4\pi\int_0^{\pi}\frac{\sin x}{(1+\cos^2 x)}\,dx$$

Applying

$$\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \text{ if } f(2a-x) = f(x)\}$$ [1]

$$2\mathrm{I} = 4\pi.2\int_0^{\frac{\pi}{2}}\frac{\sin x}{(1+\cos^2 x)}\,dx$$

$$\mathrm{I} = 4\pi\int_0^{\frac{\pi}{2}}\frac{\sin x}{(1+\cos^2 x)}\,dx$$

Put $\cos x = t \quad \Rightarrow -\sin x\,dx = dt$

As well for $x = 0,\ t = 1$ and for $x = \frac{\pi}{2},\ t = 0$

$$I = 4\pi\int_1^0 -\frac{dt}{1+t^2}$$

Applying $\int_a^b f(x)\,dx = -\int_a^b f(x)dx$ [1]

$$I = -4\pi\left[\tan^{-1} t\right]_1^0$$

$$I = -4\pi\left[\tan^{-1} 1\right]$$

$$=4\pi \times \frac{\pi}{4}$$

$$= 4\pi \times \frac{\pi}{4} = \pi^2$$ [1]

14. $\int_0^{\frac{3}{2}} x\cos\pi x dx - \int_{\frac{1}{2}}^{\frac{3}{2}} x cos\pi x dx$ [1]

$$= \left[\frac{xsin\pi x}{\pi}\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{sin\pi x}{\pi} dx - \left[\frac{xsin\pi x}{\pi}\right]_{\frac{1}{2}}^{\frac{3}{2}} + \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{-sin\pi x}{\pi} dx$$

[1]

$$= \frac{1}{2\pi} + \frac{1}{\pi^2}\left[cos\pi x\right]_0^{\frac{1}{2}} + \frac{3}{2\pi} + \frac{1}{2\pi} + \frac{1}{\pi^2}\left[cos\pi x\right]_{\frac{1}{2}}^{\frac{3}{2}}$$ [1]

$$= \frac{1}{2\pi} - \frac{1}{\pi} + \frac{3}{2\pi} + \frac{1}{2\pi} + 0$$

$$= \frac{5}{2\pi} - \frac{1}{\pi^2}$$ [1]

15. $I = \int_{-2}^{2} \frac{x^2}{1+5^x} dx = \int_0^2 \left(\frac{x^2}{1+5^x} + \frac{(-x)^2}{1+5^{-x}}\right) dx$ [1]

$$= \int_0^2 \left(\frac{x^2}{1+5^x} + \frac{x^2 5^x}{5^x+1}\right) dx = \int_0^2 x^2 \frac{\left(1+5^x\right)}{5^x+1} dx$$ [2]

$$= \int_0^2 x^2\,dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3} - 0 = \frac{8}{3}$$ [1]

16. Let $I = \int_0^{\pi} \frac{x\tan x}{\sec x + \tan x} dx$ ...(1)

Using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I = \int_0^{\pi} \left\{\frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}\right\} dx$$ [1]

$$\therefore\ I = \int_0^{\pi} \left\{\frac{-(\pi - x)\tan x}{\sec x + \tan x}\right\} dx \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{\pi\tan x}{\sec x + \tan x} dx$$ [1]

$$\therefore\ 2I = \pi\int_0^{\pi} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}} dx$$

$$\therefore\ 2I = \pi\int_0^{\pi} \frac{\sin x + 1 - 1}{1+\sin x} dx$$

$$\therefore\ 2I = \pi\int_0^{\pi} 1dx - \pi\int_0^{\pi} \frac{1}{1+\sin x} dx$$ [1]

$$\therefore\ 2I = \pi[x]_0^{\pi} - \pi\int_0^{\pi} \frac{1-\sin x}{\cos^2 x} dx$$

$$\therefore\ 2I = \pi^2 - \pi[\tan x - \sec x]_0^{\pi}$$

$$\therefore\ 2I = \pi^2[\tan\pi - \sec\pi - \tan 0 + \sec 0]$$

$$\therefore\ 2I = \pi^2 - \pi[0 - -1 - 0 + 1]$$

$$\therefore\ 2I = \pi^2 - 2\pi$$

$$\therefore\ 2I = \pi(\pi - 2)$$

$$\therefore\ I = \frac{\pi}{2}(\pi - 2)$$ [1]

17. Let $I = \int_{-\pi}^{\pi} (\cos ax - \sin bx)^2\, dx$

$$I = \int_{-\pi}^{\pi} \left(\cos^2 ax + \sin^2 bx - 2\cos ax\sin bx\right) dx$$ [1]

$$I = \int_{-\pi}^{\pi} \left(\cos^2 ax + \sin^2 bx\right) dx - 2\int_{-\pi}^{\pi} \cos ax\sin bx\, dx$$

Let $f(x) = \cos^2 ax + \sin^2 bx$

Put $x = -x$

$f(-x) = \cos^2(-ax) + \sin^2(-bx)$ [1]

$= \cos^2 ax + \sin^2 bx$ $[\because f(x) = f(-x)]$

$\Rightarrow$ $f(x)$ is even function

Let $f(x) = \cos(ax)\sin bx$

Put $x = -x$

$f(-x) = \cos(-ax)\sin(-bx)$

$= -\cos ax \sin bx = -f(x)$ (odd function) [1]

$$I = 2\int_0^{\pi}\left(\cos^2 ax + \sin^2 bx\right) dx - 0$$

$$I = 2\int_0^{\pi}\left(\frac{1+\cos 2ax}{2} + \frac{1-\cos 2bx}{2}\right) dx$$

$$I = \int_0^{\pi}\left(2 + \cos 2ax - \cos 2bx\right) dx$$

$$I = \left[2x + \frac{\sin 2ax}{2a} - \frac{\sin 2bx}{2b}\right]_0^{\pi}$$

$$I = \left(2\pi + \frac{\sin 2a\pi}{2a} - \frac{\sin 2b\pi}{2b}\right) - \left(2(0) + \frac{\sin 0}{2a} - \frac{\sin 0}{2b}\right)$$

$$I = 2\pi + \frac{\sin 2a\pi}{2a} - \frac{\sin 2b\pi}{2b} = 2\pi$$ [1]

18. Let $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\frac{\cos x}{\sin x}}}$$ ...(i)

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$$ [1]

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}+\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}} dx$$ [1]

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}} dx$$ [1]

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$$ ...(ii) [1]

Adding equation (i) and (ii)

$$2I = \int_{\pi/6}^{\pi/3} \left(\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\right) dx$$ [1]

$$2I = \int_{\pi/6}^{\pi/3} 1.dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$\therefore I = \frac{\pi}{12}$ [1]

19. $f(x) = 2x^2 + 5x, b = 3, a = 1, nh = 3 - 1 = 2$

$f(a) = f(1) = 2(1)^2 + 5(1) = 7$

$f(a+h) = f(1+h) = 2(1+h)^2 + 5(1+h)$

$= 2(1 + h^2 + 2h) + 5 + 5h$

$= 2 + 2h^2 + 4h + 5 + 5h$

$= 2h^2 + 9h + 7$ [1]

$f(a+2h) = f(1+2h) = 2(1+2h)^2 + 5(1+2h)$

$= 2(1 + 4h^2 + 4h) + 5 + 10h$

$= 2 + 8h^2 + 8h + 5 + 10h$

$= 8h^2 + 18h + 7$ [1]

$f(a+(n-1)h) = f(1+(n-1)h)$

$= 2\left[1+(n-1)h^2\right] + 5\left[1+(n-1)h\right]$

$= 2 + 2(n-1)^2 h^2 + 4(n-1)h + 5 + 5(n-1)h$

$= 2(n-1)^2 h^2 + 9(n-1)h + 7$ [1]

$$\int_a^b (x)\, dx = \lim_{h\to 0} h\begin{bmatrix} f(a) + f(a+h) + f(a+2h) + \\ f(a-(n-1)h) \end{bmatrix}$$

$$\int_1^3 (2x^2+5x)dx = \lim_{h\to 0} h\left[7 + \left(2h^2+9h+7\right)\right] +$$

$$\left(8h^2+18h+7\right) + ... + \left(2(n-1)^2 h^2 + 9(n-1)h + 7\right)$$

[1]

$$= \lim_{h\to 0} h\left[7n+2h^2\left(1^1+2^2+...+(n-1)^2\right)\right.$$
$$\left.+9h(1+2+...+(n-1))\right)$$

$$= \lim_{h\to 0} h\left[7n+2h^2\frac{n(n-1)(2n-1)}{6}+\frac{9hn(n-1)}{2}\right]$$

$$= \lim_{h\to 0}\left[7hn+\frac{nh(nh-h)(2nh-h)}{3}+\frac{9}{2}nh(nh-h)\right]$$

$$= \lim_{h\to 0}\left[7(2)+\frac{2}{3}(2-n)(4-h)+\frac{9}{2}.2(2-h)\right] \quad [1]$$

$[nh=2]$

$$= 14+\frac{2}{3}(2-0)(4-0)+9(2-0)$$

$$= 14+\frac{16}{3}+18 = 32+\frac{16}{3} = \frac{96+16}{3} = \frac{112}{3} \quad [1]$$

20. Let I = $\int_0^{\pi/2} 2\sin x\cos x\tan^{-1}(\sin x)\,dx$

$Let\ p = \sin x$

$dp = \cos x dx$ [1]

when $x = \frac{\pi}{2}, p = 1$

when $x = 0, p = 0$ [1]

$$= 2\int_0^1 p\tan^{-1}(p)dp \quad [1]$$

Applying by parts

$$= 2\left\{\left[\tan^{-1} p.\frac{p^2}{2}\right]_0^1 - \int_0^1 \frac{1}{1+p^2}\times\frac{p^2}{2}\,dx\right\}$$

$$= 2\left\{\left[\frac{1}{2}\tan^1(1)\right]-[0]\right\}-2\times\frac{1}{2}\int_0^1\frac{\left[\left(P^2+1\right)-1\right.}{1+P^2}\,dp \quad [1]$$

$$= 2\left[\frac{1}{2}\times\frac{\pi}{4}\right]-\int_0^1\left(1-\frac{1}{1+p^2}\right)dp$$

$$= \frac{\pi}{4}-\left[p-\tan^{-1}(p)\right]_0^1 \quad [1]$$

$$= \frac{\pi}{4}-\left(1-\tan^{-1}(1)\right)+\left(0-\tan^{-1}(0)\right)$$

$$= \frac{\pi}{4}-1+\frac{\pi}{4} = \frac{\pi}{2}-1 \quad [1]$$

21. Let $I = \int_0^{\frac{\pi}{2}} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}dx$

$$\left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right).\cos\left(\frac{\pi}{2}-x\right)}{\sin^4 x+\cos^4\left(\frac{\pi}{2}-x\right)}\,dx \quad [1]$$

$$\text{I} = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right)\cos x\sin x}{\sin^4 x+\cos^4 x}dx \quad [1]$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{x\sin x\cos x+\left(\frac{\pi}{2}-x\right)\cos x\sin x}{\sin^4 x+\cos^4 x}dx$$

$$2I = \int_0^{\pi/2} \frac{\frac{\pi}{2}\sin x\cos x}{\sin^4 x\cos^4 x} \quad [1]$$

Dividing numerator and denominator by $\cos^4 x$, we get

$Let\ p = \tan^2 x$

$dp = 2\tan x\sec^2 x\,dx$

$\frac{dp}{2} = \tan x\sec^2 x\,dx$ [1]

when $x = 0, p = 0$

when $x = \frac{\pi}{2}, p = \infty$

$$2I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\tan x.\sec^2 x}{\tan^4 x+1}dx$$

$$= \frac{\pi}{2}\times\frac{1}{2}\int_0^{\infty}\frac{dp}{p^2+1}$$

$$= \frac{\pi}{4}\left[\tan^{-1}(p)\right]_0^{\infty} \quad [1]$$

$$2I = \frac{\pi}{4}\left[\tan^{-1}(\infty)-\tan^{-1}(0)\right]$$

$$\text{I} = \frac{\pi}{8}\left[\frac{\pi}{2}-0\right] = \frac{\pi^2}{16} \quad [1]$$

22. $\int_1^4 \left(x^2 - x\right) dx$

Let $I = \int_1^4 \left(x^2 - x\right) dx$

$$I = \int_1^4 \left(x^2\right) dx - \int_1^4 x dx$$

$I = A - B$ [1]

We know that

$$\int_a^b f(x)\,dx = \lim_{h\to 0}\left[\begin{array}{l} f(a) + f(a+h) + f(a+2h) \\ + \ldots + f\{a+(n-1)h\}\end{array}\right]$$

Where $h = \dfrac{b-a}{n}$

$\therefore h = \dfrac{3}{n}$ [1]

$$A = \int_1^4 \left(x^2\right) dx$$

$$= \lim_{h\to 0} h\left[f(1) + f\left(1+\frac{3}{n}\right) + \ldots + f\left\{1+(n-1)\frac{3}{n}\right\}\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[1^2 + \left\{\begin{array}{l}1^2 + \left(\frac{3}{n}\right)^2 \\ +2.\frac{3}{n}\end{array}\right\} + \ldots + \left\{\begin{array}{l}1^2 + \left(\frac{3(n-1)^2}{n}\right) \\ +2.3.\left(\frac{n-1}{n}\right)\end{array}\right\}\right]$$

$$= 3\lim_{n\to\infty}\left(\frac{1}{n}\right)\left[\begin{array}{l} n + \left(\frac{3}{n}\right)^2\left\{1^2 + 2^2 + \ldots(n-1)^2\right\} \\ +2.\frac{3}{n}(1+2+\ldots n-1)\end{array}\right]$$

$$= 3\lim_{n\to\infty}\left(\frac{1}{n}\right)\left[n + \frac{9}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{6}{n}\left(\frac{n(n-1)}{2}\right)\right]$$

[1]

$$= 3\lim_{n\to\infty}\left(\frac{1}{n}\right)\left[n + \frac{3(n-1)(2n-1)}{2n} + \left(\frac{6(n-1)}{2}\right)\right]$$

$$= 3\lim_{n\to\infty}\left(\frac{1}{n}\right)\left[n + \frac{6n^2 - 9n + 3}{2n} + 3n - 3\right]$$

$$= 3\lim_{n\to\infty}\left(\frac{1}{n}\right)\left[n + \frac{6n^2 - 9n + 3}{2n} + 3n - 3\right]$$

$$= 3\lim_{n\to\infty}\left[1 + 3 - \frac{9}{2n} + \frac{3}{2n^2} + 3 - \frac{3}{n}\right]$$

$$= 3[1+3+3]$$

$= 21$ [1]

Now,

$$B = \int_1^4 x dx$$

$$= \lim_{n\to\infty} h\left[1 + \left(1+\frac{3}{n}\right) + \ldots\left\{1 + \frac{(n-1)3}{n}\right\}\right]$$

$$= \lim_{n\to\infty}\left(\frac{3}{n}\right)\left[1+1+\ldots n\ times + \frac{3}{n}(1+2+3+\ldots n-1)\right]$$

$$= 3\lim_{n\to\infty}\left(\frac{1}{n}\right)\left[n + \frac{3}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$= 3\lim_{n\to\infty}\left(\frac{1}{n}\right)\left[n + \frac{3(n-1)}{2}\right]$$ [1]

$$= 3\lim_{n\to\infty}\left[1 + \frac{3(n-1)}{2n}\right]$$

$$= 3\lim_{n\to\infty}\left[1 + \frac{3}{2} - \frac{1}{2n}\right]$$

$$= 3\left[1 + \frac{3}{2}\right]$$

$$= \frac{15}{2}$$

Thus,

$I = A + B$

$$I = 21 - \frac{15}{2}$$

$I = \dfrac{27}{2}$ [1]

23. Let $I = \int_0^\pi \log(1+\cos x)\,dx$ ...(1)

As we know that

$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ [1]

Now,

$$I=\int_0^{\pi}\log\left(1+\cos\left(\pi-x\right)\right)dx$$

$$I=\int_0^{\frac{\pi}{2}}\log\left(1-\cos x\right)dx \qquad ...(2) \qquad [1]$$

Adding (1) and(2), we obtain

$$2I=\int_0^{\pi}\left\{\log\left(1-\cos x\right)+\log\left(1-cosx\right)\right\}dx$$

$$2I=\int_0^{\pi}\log\left(1-\cos^2 x\right)dx$$

$$2I=\int_0^{\pi}\log\left(\sin^2 x\right)dx_9$$

$$2I=2\int_0^{\pi}\log sinxdx$$

$$I=\int_0^{\pi}\log\sin xdx$$

$$\sin\left(\pi-x\right)=\sin x$$

$$I=2\int_0^{\frac{\pi}{2}}\log\sin x \qquad ...(3) \qquad [1]$$

$$I=2\int_0^{\frac{\pi}{2}}\log\sin\left(\frac{\pi}{2}-x\right)dx$$

$$I=2\int_0^{\frac{\pi}{2}}\log\cos xdx \qquad ...(4) \qquad [1]$$

Adding (3) and (4), we obtain

$$I=2\int_0^{\frac{\pi}{2}}\log\sin x+\log\cos xdx$$

$$I=2\int_0^{\frac{\pi}{2}}\log\sin x+\log\cos x+\log 2-\log 2 \qquad [1]$$

$$I=2\int_0^{\frac{\pi}{2}}\log\left(2\sin xcosx\right)-\log 2$$

$$I=2\int_0^{\frac{\pi}{2}}\log\sin 2x-\log 2$$

$$I=-\pi\log 2 \qquad [1]$$

24. L.H.S $=\int_0^{\frac{\pi}{4}}\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx$

$$=\int_0^{\frac{\pi}{4}}\left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}}+\frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right)dx$$

$$=\int_0^{\frac{\pi}{4}}\left(\frac{(\sin x+\cos x)}{\sqrt{\cos x.\sin x}}\right)dx \qquad [1]$$

$$=\sqrt{2}\int_0^{\frac{\pi}{4}}\frac{(\sin x+\cos x)}{\sqrt{1-(\sin x-\cos x)^2}}dx \qquad [1]$$

Let $\sin x-\ \cos x=z$ [1]

$\therefore\ \left(\cos x+\ \sin x\right)dx=dz$

Also if $x=\ 0, z=\ -1$

and $x=\frac{\pi}{4}, z=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=0$ [1]

Thus, LHS $=\sqrt{2}\int_{-1}^{0}\frac{dz}{\sqrt{1-z^2}}$

$$=\sqrt{2}\left[\sin^{-1}z\right]_{-1}^{0}$$

$$=\sqrt{2}\left[\sin^{-1}0-\sin^{-1}\left(-1\right)\right] \qquad [1]$$

$$=\sqrt{2}\left[0-\left(\frac{-\pi}{2}\right)\right]$$

$$=\sqrt{2}.\frac{\pi}{2} \qquad [1]$$

25. $I=\int_0^{\frac{\pi}{2}}\frac{2^{sinx}}{x^{sinx}+2^{cosx}}dx$ ...(1)

Using $\int_0^a(x)\,dx=\int_0^a f(a-x)\,dx$ [1]

$$=\int_0^{\frac{\pi}{2}}\frac{2^{\sin\left(\frac{\pi}{2}-x\right)}}{2^{\sin\left(\frac{\pi}{2}-x\right)}+2^{\cos\left(\frac{\pi}{2}-x\right)}}dx$$ [1]

$$=\int_0^{\frac{\pi}{2}}\frac{2^{cosx}}{2^{\sin x}+2^{\cos x}}dx \quad ...(2)$$ [1]

Adding (1) and (2) we get

$$2I=\int_0^{\frac{\pi}{2}}1\,dx=[x]_0^{\frac{\pi}{2}}=\frac{\pi}{2}$$ [2]

$$\Rightarrow \text{I}=\frac{\pi}{4}$$ [1]

26. Given $\int_0^{\frac{\pi}{4}}\frac{\sin x+\cos x}{16+9\sin 2x}dx$

$$=\int_0^{\frac{\pi}{4}}\frac{\sin x+\cos x}{16+9\sin 2x+9-9}dx$$ [1]

$$=\int_0^{\frac{\pi}{4}}\frac{\sin x+\cos x}{25-9(1-\sin 2x)}dx$$

Using the identity

$\sin 2x=\sin x\cos x,\quad \sin^2 x+\cos^2 x=1$ [1]

$$=\int_0^{\frac{\pi}{4}}\frac{\sin x+\cos x}{25-9\left(\sin^2 x+\cos^2 x-2\sin x\cos x\right)}dx$$

$$=\int_0^{\frac{\pi}{4}}\frac{\sin x+\cos x}{25-9(\sin x-\cos x)^2}dx$$

Let $t=\sin x-\cos x$

$$\frac{dt}{dx}=\cos x+\sin x$$

$dt=(\cos x+\sin x)\,dx$ ...(1) [1]

When $x=0$ then $t=-1$.

When $x=\frac{\pi}{4}$ then $t=0$.

$$=\int_{-1}^{0}\frac{dt}{25-9(t)^2}$$

$$=\frac{1}{9}\int_{-1}^{0}\frac{dt}{\left(\frac{25}{9}\right)-(t)^2}$$

$$=\frac{1}{9}\int_{-1}^{0}\frac{dt}{\left(\frac{5}{3}\right)^2-(t)^2}$$

Using $\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$

$$=\frac{1}{9}\left[\frac{1}{2\times\frac{5}{3}}\log\left|\frac{\frac{5}{3}+t}{\frac{5}{3}-t}\right|\right]_{-1}^{0}$$ [1]

$$=\frac{1}{30}\left[\log\left|\frac{5+3t}{5-3t}\right|\right]_{-1}^{0}$$

$$=\frac{1}{30}\left[\log\left|\frac{5+3(0)}{5-3(0)}\right|-\log\left|\frac{5+3(-1)}{5-3(-1)}\right|\right]$$

$$=\frac{1}{30}\left[\log\left|\frac{5}{5}\right|-\log\left|\frac{2}{8}\right|\right]$$

$$=\frac{1}{30}\left[\log|1|-\log\left|\frac{1}{4}\right|\right]$$

$$=\frac{1}{30}\left[\log|1|-\log|1|+\log|4|\right]$$ [1]

Using $\log 1=0$ and $\log\frac{A}{B}=\log A-\log B$.

$$=\frac{1}{30}\log|4|$$

$$=\frac{1}{30}\log\left|2^2\right|$$

Using $\log a^b = b\log a$

$= \frac{2}{30}\log|2|$

$= \frac{1}{15}\log|2|$ [1]

27. Given $\int_1^3 \left(x^2 + 3x + e^x\right)dx$

$f(x) = x^2 + 3x + e^x, a = 1, \ b = 3$

$nh = b - a = 3 - 1 = 2$ [1]

$f(a) = f(1) = (1)^2 + 3(1) + e^1 = 4 + e$

$$\begin{aligned} f(a+h) &= f(1+h) \\ &= (1+h)^2 + 3(1+h) + e^{1+h} \\ &= 1 + h^2 + 2h + 3 + 3h + e^{1+h} \\ &= h^2 + 5h + 4 + e.e^h \end{aligned}$$ [1]

$$\begin{aligned} f(a+2h) &= f(1+2h) \\ &= (1+2h)^2 + 3(1+2h) + e^{1+2h} \\ &= 1 + 4h^2 + 4h + 3 + 6h + e^{1+2h} \\ &= 4h^2 + 10h + 4 + e.e^{2h} \end{aligned}$$ [1]

$f(a+(n-1)h) = f(1+(n-1)h)$

$= (1+(n-1)h)^2 + 3(1+(n-1)h) + e^{1+(n-1)h}$

$= 1 + (n-1)^2 h^2 + 2(n-1)h + 3 + 3(n-1)h + e^{1+(n-1)h}$

$= (n-1)^2 h^2 + 5(n-1)h + 4 + e.e^{(n-1)h}$

$$\int_1^3 \left(x^2 + 3x + e^x\right)dx = \lim_{h\to 0} h\begin{bmatrix} f(a) + f(a+h) + f(a+2h) \\ +\dots + f(a+(n-1)h) \end{bmatrix}$$

$$= \lim_{h\to 0} h\begin{bmatrix} (4+e) + \left(h^2 + 5h + 4 + e.e^h\right) + \\ \left(4h^2 + 10h + 4 + e.e^{2h}\right)\dots + \\ \left((n-1)^2 h^2 + 5(n-1)h + 4 + e.e^{(n-1)h}\right) \end{bmatrix}$$

[1]

$$= \lim_{h\to 0} h\begin{bmatrix} h^2\left(1^2 + 2^2 \dots\dots (n-1)^2\right) + \\ 5h(1 + 2 + \dots + (n-1)) + \\ 4(1 + 1\dots + 1) + e\left(1 + e^h + e^{2h} + \dots e^{(n-1)h}\right) \end{bmatrix}$$

Using $S_n = \frac{A\left(R^n - 1\right)}{(R-1)}$

$$= \lim_{h\to 0}\begin{bmatrix} h^3 \frac{n(n-1)(2n-1)}{6} + \frac{5h^2 n(n-1)}{2} \\ +4nh + eh\frac{e^{nh}-1}{e^h - 1} \end{bmatrix}$$

$$= \lim_{h\to 0}\begin{bmatrix} \frac{nh(nh-h)(2nh-h)}{6} + \frac{5hn(nh-h)}{2} \\ +4nh + eh\frac{e^{nh}-1}{\frac{\left(e^h-1\right)}{h}\times h} \end{bmatrix}$$ [1]

As nh = 2

$$= \lim_{h\to 0}\left[\frac{2(2-h)(4-h)}{6} + \frac{10(2-h)}{2} + 8 + e\frac{e^2 - 1}{\frac{\left(e^h - 1\right)}{h}}\right]$$

Using $\lim_{h\to 0}\frac{e^h - 1}{h} = 1$

$$= \left[\frac{2(2-0)(4-0)}{6} + \frac{10(2-0)}{2} + 8 + e\frac{\left(e^2 - 1\right)}{1}\right]$$

$$= \left[\frac{8}{3} + 18 + e\left(e^2 - 1\right)\right]$$

$$= \left[\frac{62}{3} + e^3 - e\right]$$ [1]

Smart Notes ....

Smart Notes ....

CHAPTER 8

Application of Integrals

Chapter Analysis of Previous Three Years' Board Exams

Number of Questions Asked in Exams

Topics	**2018**		**2017**		**2016**	
	Delhi	**All India**	**Delhi**	**All India**	**Delhi**	**All India**
Area Under Curve	6 marks	6 marks	6 marks	6 marks	6 marks	6 marks

Summary

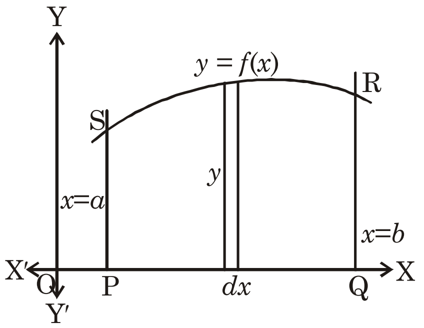
- **Area under simple curves**
  - Consider that a curve y = f(x), the line x = a, x = b and x-axis collectively acquires an area and the area under the curve is considered as composed of large number of vertical thin strips.

    Now assume that there is an arbitrary strip with height y and width dx.

    Then dA which represents area of elementary strip = ydx, where y = f(x).

    Total area A of the region between the curve y = f(x), x = a, x = b and x-axis is equal to the sum of areas of all elementary vertical thin strips across the region PQRS.

    which is given by, $A = \int_a^b y\,dx = \int_a^b f(x)\,dx$



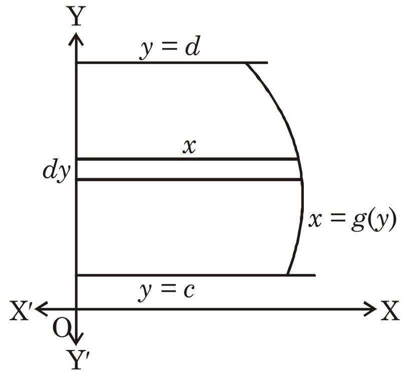
    Elementary Area: The area which is located at an arbitrary position within the region which is specified by some value of x between a and b

  - Now consider the area A of the region which is bounded by the curve x = g(y), the lines y = c, y = d and y-axis.

    Total area A of the region between the curve x = g(y), y = c, y = d and y-axis is equal to the sum of areas of all elementary horizontal thin strips.

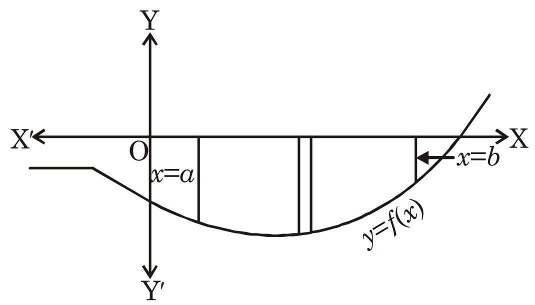
    In this case the area A is given by

$$A = \int_c^d x\,dy = \int_c^d g(y)dy$$



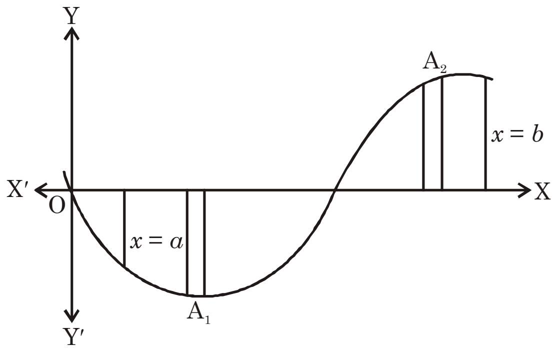
  - If the curve is positioned below x-axis, which is f(x) < 0 from x = a to x = b, then the numerical value of the area which is bounded by the curve y = f(x), x-axis and the ordinates x = a, x = b will come out to be negative. But, if the numerical value of the area is to be taken into consideration, then is given by:

$$A = \left|\int_a^b f(x)\,dx\right|$$



  - There is a possibilitythat some portion of the curve is located above x-axis and some portion of it is located below x-axis.

    Let suppose A_1 is the area below x-axis and $A_2$is the area above x-axis. Now, the area of the region which is bounded by the curve y = f(x), x = a, x = b and x-axis can be given by $A = |A_1| + |A_2|$.



- **Area of the region bounded by a curve and a line**
  - ➢ Area of the region bounded by a line and a curve is used to find the area bounded by a line and a parabola, a line and an ellipse, a line and a circle etc. The standard equation will be used for these mentioned curves.
  - ➢ Area of the region can be calculated by taking the sum of the area of either horizontal or vertical elementary strips but vertical strips are mostly preferred.
- **Area between two curves**
  - ➢ Assume that there are two curves, y = f(x) and y = g(x), where f(x) ≥ g(x) in [a, b]. The ordinates x = a and x = b give the point of intersection of these two curves. Suppose that these curves intersect at f(x) with width dx.

    Consider an elementary vertical strip of height y, where y = f(x).

    $\therefore \quad dA = y\,dx$

    Now the area is given by,

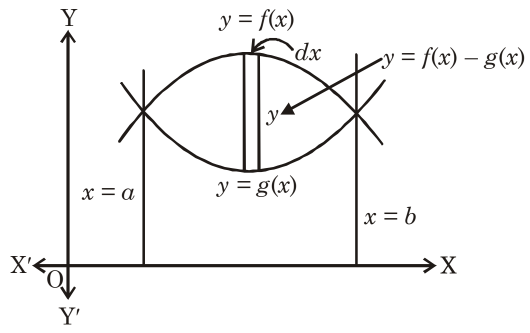
    $$A = \int_a^b [f(x) - g(x)]dx = \int_a^b f(x)\,dx - \int_a^b g(x)dx$$

    which can be stated as,

    A = Area bounded by the curve {y = f(x)}

    – Area bounded by the curve {y = g(x)}

    where f(x) > g(x).



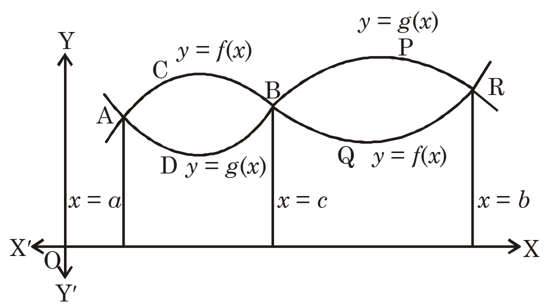
- ➢ In other case if the two curves y = f(x) and y = g(x) where f(x) ≤ g(x) in [a, c] and f(x) ≤ g(x) in [c, b] with a condition that a < c < b, intersect at x = a, x = c and x = b, then the area bounded by the curves is given by:

$$A = \int_a^c |f(x) - g(x)|dx + \int_c^b |g(x) - f(x)|dx$$

which is stated as:

Total area = Area of the region ACBDA

+ Area of the region BPRQB



PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

6 Marks Questions

1. Using integration, find the area of region bounded by the triangle whose vertices are (–2, 1), (0, 4) and (2, 3).

[DELHI 2017]

2. Find the area bounded by the circle $x^2 + y^2 = 16$ and the line $\sqrt{3}y = x$ in the first quadrant, using integration.

[DELHI 2017]

3. Using integration find the area of the region

$$\left\{(x, y) : x^2 + y^2 \le 2ax, y^2 \ge ax, x, y \ge 0\right\}$$

[DELHI 2016]

4. Using integration ,find the area of the region enclosed between the two circle $x^2 + y^2 = 4 \; and (x-2)^2 + y^2 = 4.$

[2013]

5. Using the method of integration find the area of the region bounded by the lines $3x-2y+1=0$, $2x+3y-21=0$ and $x-5y+9=0$.

[ALL INDIA 2012]

6. Using integration find the area of the triangular region whose sides have the equations $y = 2x+1, y = 3x+1$ and $x = 4$

[DELHI 2011]

7. Using the method of integration , find the area of the lines $3x - 2y + 1 = 0$, $2x + 3y - 21 = 0$ and $x - 5y + 9 = 0$.

[DELHI 2012]

8. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

[ALL INDIA 2013]

9. Using integration, find the area of the region bounded by the triangle whose vertices are (–1, 2), (1, 5) and (3, 4).

[ALL INDIA 2014]

10. Using integration, find the area of the region bounded by the lines y = 2 + x, y = 2 – x and x = 2

[ALL INDIA 2015]

11. Using the method of integration, find the area of the triangular region whose vertices (2, – 2), (4, 3) are and (1, 2).

[ALL INDIA 2016]

12. Using the method of integration, find the area of the triangle ABC, coordinates of whose vertices are A (4, 1), B (6, 6) and C (8, 4).

[ALL INDIA 2017]

13. Find the area enclosed between the parabola $4y = 3x^2$ and the straight line $3x - 2y + 12 = 0$.

[ALL INDIA 2017]

14. Using integration, find the area of the region in the first quadrant enclosed by x-axis, line $y = x$ and the circle $x^2 + y^2 = 32$.

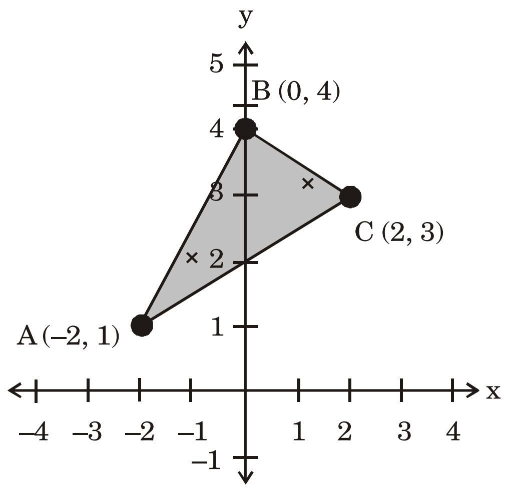
[DELHI 2018]

15. Using Integration find the area of the triangle formed by positive x-axis and tangent and normal to the circle $x^2 + y^2 = 4$ at $\left(1, \sqrt{3}\right)$.

[DELHI 2015]

Solutions

1.



[1]

Let $A(-2,1), B(0,4)$ and $C(2,3)$

Equation of the line AB,

$$(y-1) = \frac{4-1}{0+2}(x+2) \Rightarrow 2(y-1) = 3(x+2) \quad [1]$$

$$2y - 3x = 8 \Rightarrow y = \frac{3x+8}{2} \quad ....(1)$$

Equation of the line BC,

$$(y-4) = \frac{3-4}{2-0}(x-0) \Rightarrow 2(y-4) = -1(x)$$

$$2y + x = 8 \Rightarrow y = \frac{8-x}{2} \quad ....(2) \quad [1]$$

Equation of the line AC,

$$(y-1)=\frac{3-1}{2+2}(x+2)\Rightarrow 4(y-1)=2(x+2)$$

$$4y-2x=8\Rightarrow y=\frac{8+2x}{4}=\frac{4+x}{2}\quad ....(3)\qquad [1]$$

Area of the shaded region,

$$=\int_{-2}^{0}\left(\frac{8+3x}{2}\right)dx+\int_{0}^{2}\left(\frac{8-x}{2}\right)dx-\int_{-2}^{2}\left(\frac{4+x}{2}\right)dx$$

$$=\frac{1}{2}\left[8x+\frac{3x^2}{2}\right]_{-2}^{0}+\frac{1}{2}\left[8x-\frac{x^2}{2}\right]_{0}^{2}-\frac{1}{2}\left[4x+\frac{x^2}{2}\right]_{-2}^{2}$$

[1]

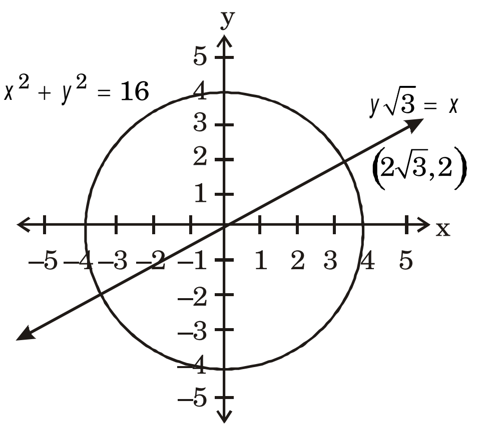
$$=\frac{1}{2}[(0)-(-16+6)]+\frac{1}{2}[(16-2)-0]$$

$$-\frac{1}{2}[(8+2)-(-8+2)]$$

$$=5+7-8$$

$= 4$ sq. units [1]

2.



[1]

$$x^2+y^2=16$$

$$y^2=16-x^2$$

$$y=\pm\sqrt{4^2-x^2}\qquad [1]$$

x	0	±4
y	±4	0

Similarly, $\sqrt{3}y=x$

$$y=\frac{x}{\sqrt{3}}$$

x	0	$\sqrt{3}$	3
y	0	1	$\sqrt{3}$

[1]

Points of intersection,

$$x^2+y^2=16,\ y=\frac{x}{\sqrt{3}}$$

$$x^2+\left(\frac{x}{\sqrt{3}}\right)^2=16$$

$$x^2+\frac{x^2}{3}=16$$

$$\frac{3x^2+x^2}{3}=\frac{16}{1}$$

$$4x^2=48\Rightarrow x^2=\pm 12$$

$$\Rightarrow x=\pm\sqrt{12}=\pm\sqrt{4\times 3}=\pm 2\sqrt{3}\qquad [1]$$

Area of the shaded region

$$=\int_{0}^{2\sqrt{3}}\frac{x}{\sqrt{3}}dx+\int_{2\sqrt{3}}^{4}\sqrt{4^2-x^2}\,dx$$

$$=\frac{1}{\sqrt{3}}\times\frac{1}{2}\left(x^2\right)_{0}^{2\sqrt{3}}+\frac{1}{2}\left(x\sqrt{16-x^2}+16\sin^{-1}\frac{x}{4}\right)_{2\sqrt{3}}^{4}$$

[1]

$$\left(\because\int\sqrt{a^2-x^2}\,dx=\frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{4}\right)+c\right)$$

$$=\frac{1}{2\sqrt{3}}(12)+\frac{1}{2}\left(\begin{array}{l}\left(0+16\sin^{-1}1\right)-\\\left(2\sqrt{3}\times 2+16\sin^{-1}\frac{2\sqrt{3}}{4}\right)\end{array}\right)$$

$$=\frac{6}{\sqrt{3}}+\frac{1}{2}\left(16\times\frac{\pi}{2}-4\sqrt{3}-16\times\frac{\pi}{3}\right)$$

$$=\frac{6}{\sqrt{3}}+\frac{1}{2}\left(16\times\frac{\pi}{2}-4\sqrt{3}-16\times\frac{\pi}{3}\right)$$

$$=\frac{6}{\sqrt{3}}+4\pi-2\sqrt{3}-\frac{8\pi}{3}$$

$$=\left(\frac{6}{\sqrt{3}}-2\sqrt{3}\right)+\left(4-\frac{8}{3}\right)\pi$$

$$=\left(\frac{6-6}{\sqrt{3}}\right)+\left(\frac{12-8}{3}\right)\pi$$

$$=\frac{4\pi}{3}\text{sq. units}\qquad [1]$$

3. Consider $x^2+y^2\le 2ax$.

Let, $x^2+y^2=2ax$

$\Rightarrow y^2 = 2ax - x^2$

$\Rightarrow y = \pm\sqrt{2ax - x^2}$

Draw the table for the values of x and y we get;

x	0	a	2a
Y	0	±a	0

[1]

Now consider $y^2 \geq ax$

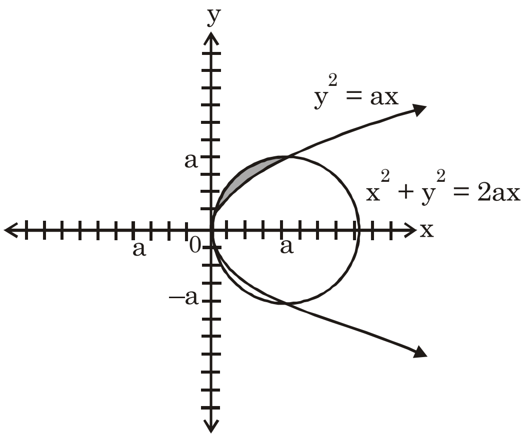
Let, $y^2 = ax$

$\Rightarrow y = \pm\sqrt{ax}$

Draw the table for the values of x and y we get;

x	0	A
Y	0	±a

[1]



[1]

Area of the shaded region can be calculated as:

$$\int_0^a \sqrt{2ax - x^2}\,dx - \int_0^a \sqrt{ax}\,dx$$

$$\int_0^a \sqrt{-(x^2 - 2ax)}\,dx - \sqrt{a}\int_0^a x^{\frac{1}{2}}\,dx$$

$$= \int_0^a \sqrt{-[(x^2 - 2ax + a^2) - a^2]}\,dx - \sqrt{a}\int_0^a x^{\frac{1}{2}}\,dx$$

$$= \int_0^a \sqrt{a^2 - (x-a)^2}\,dx - \sqrt{a}\int_0^a x^{\frac{1}{2}}\,dx$$ [1]

We know that

$$\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right) + C$$

$$\Rightarrow \int_0^a \sqrt{a^2 - (x-a)^2}\,dx - \sqrt{a}\int_0^a x^{\frac{1}{2}}\,dx = \frac{1}{2}$$

$$\left((x-a)\sqrt{a^2 - (x-a)^2} + a^2 \sin^{-1}\frac{x-a}{a}\right)_0^a - \sqrt{a}.\frac{2}{3}[x^{\frac{3}{2}}]_0^a$$

$$= \frac{1}{2}\left\{\begin{array}{l}\left[(a-a)\sqrt{a^2 - (a-a)^2} + a^2 \sin^{-1}\frac{a-a}{a}\right] \\ -\left[(0-a)\sqrt{a^2 - (0-a)^2} + a^2 \sin^{-1}\frac{0-a}{a}\right]\end{array}\right\}$$ [1]

$$-\frac{2\sqrt{a}}{3}[a^{\frac{3}{2}} - 0]$$

$$= \frac{1}{2}\left[a^{2\sin^{-1}} 0 - a^2 \sin^{-1}(-1)\right]$$

$$= \frac{1}{2}\left[a^2 \sin^{-1}\sin(0)\right] - \frac{1}{2}\left[a^2 \sin^{-1}\sin\left(\frac{-\pi}{2}\right)\right] - \frac{2a^2}{3}$$

$$= \frac{1}{2}\left[a^2 \times 0\right] - \frac{1}{2}\left[a^2 \times \left(\frac{-\pi}{2}\right)\right] - \frac{2a^2}{3}$$

$$= a^2\left(\frac{\pi}{4} - \frac{2}{3}\right) \text{ sq. units}$$ [1]

4. Given : The given circle are

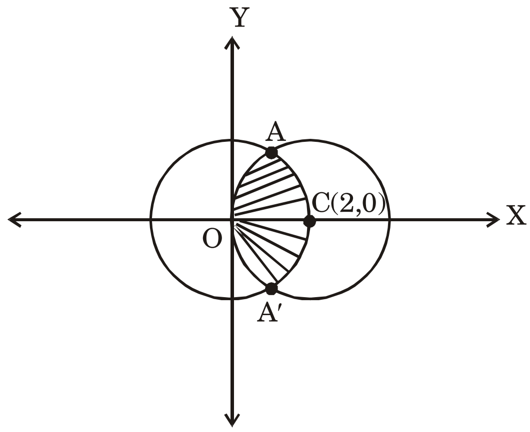
$x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$

$x^2 + y^2 = 4$

$x^2 + y^2 = 2^2$ [1]

Centre = $(0,0)$

Radius = 2



[1]

Now, $x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$y = \pm\sqrt{4 - x^2}$

x	0	±2
y	±2	0

[1]

$(x-2)^2 + y^2 = 4$

$(x-2)^2 + (y-0)^2 = (2)^2$

$(x-2)^2 + (y-0)^2 = (2)^2$

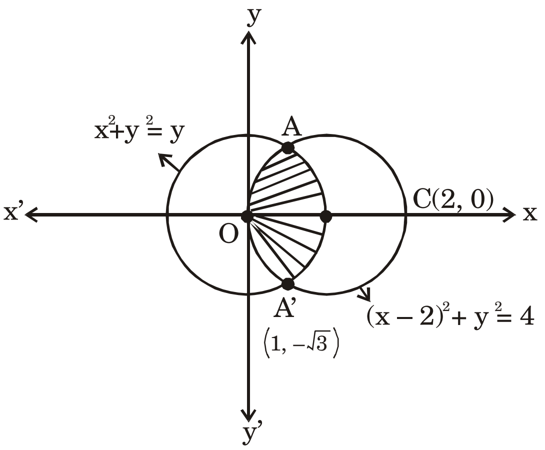
$Centre = (2,0)$

$Radius = 2$

$y^2 = 4 - (x-2)^2$

$y = \pm\sqrt{4 - (x-2)^2}$

x	0	2	4
y	0	±2	0



[1]

For point of intersection

$(x-2)^2 + y^2 = 4$

$x^2 - 4x + 4 - x^2 = 0$

$-4x = -4 \qquad : x = 1$

$$Shaded\ Area = 4\int_1^2 \sqrt{2^2 - x^2}\,dx$$

$$= 4 \times \frac{1}{2}\left[x\sqrt{4 - x^2} + 4\sin^{-1}\frac{x}{2}\right]_1^2$$

$$\left[\int\sqrt{a^2 - x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2\sin^{-1}\frac{x}{a}\right) + C\right]$$

[1]

$$\left[\begin{array}{l}\sin\frac{\pi}{2} = 1, \sin\frac{\pi}{6} = \frac{1}{2} \\ \sin^{-1}(\sin\theta) = \theta\end{array}\right]$$

$$= 2\left\{\left[2\sqrt{4-4} + 4\sin^{-1}\left(\frac{2}{2}\right)\right] - \left[1\sqrt{4-1} + 4\sin^{-1}\frac{1}{2}\right]\right.$$

$$= 2\left\{\left[0 + 4\frac{\pi}{2}\right] - \left[\sqrt{3} + 4\frac{\pi}{6}\right]\right\}$$

$$= 2\left[2\pi - \sqrt{3} - \frac{2\pi}{3}\right]$$

$$= 2\left[\frac{4\pi}{3} - \sqrt{3}\right] \text{sq. units}$$

[1]

5. $3x - 2y + 1 = 0 \qquad 2x + 3y - 21 = 0 \qquad x - 5y + 9 = 0$

$3x + 1 = 2y \qquad 3y = 21 - 2x \qquad x + 9 = 5y$

$y = \frac{3x-1}{2} \qquad y = \frac{21-2x}{3} \qquad y = \frac{x+9}{5}$

[½]

$y = \frac{3x-1}{2}$

x	3	1
y	5	2

[½]

$y = \frac{21-2x}{3}$

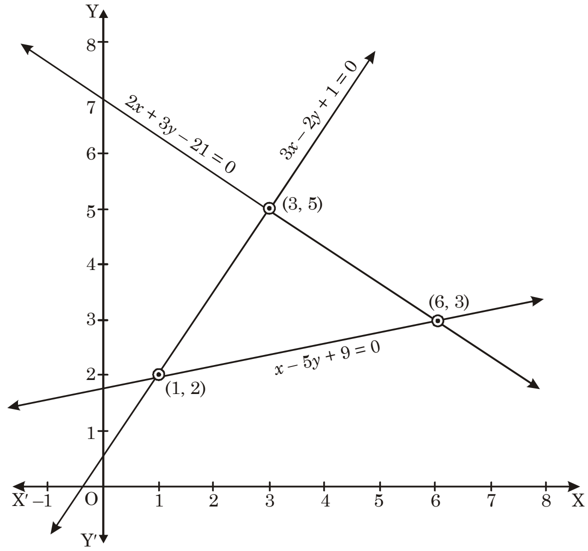
x	3	6
y	5	1

[½]

$y = \frac{x+9}{5}$

x	1	6
y	2	3

[½]



[1]

$$=\int_1^3\left(\frac{3x+1}{2}\right)dx+\int_3^6\left(\frac{21-2x}{3}\right)dx-\int_1^6\frac{x+9}{5}dx$$

$$=\frac{1}{2}\left[\frac{3x^2}{2}+x\right]_1^3+\frac{1}{3}\left[21x-\frac{2x^2}{2}\right]_3^6-\frac{1}{5}\left[\frac{x^2}{2}+9x\right]_1^6$$ [1]

$$=\frac{1}{2}\left[\left(\frac{3}{2}(9)+3\right)-\left(\frac{3}{2}+1\right)\right]+\frac{1}{3}\left[21(6)-36-\left(21(3)-9\right)\right]$$

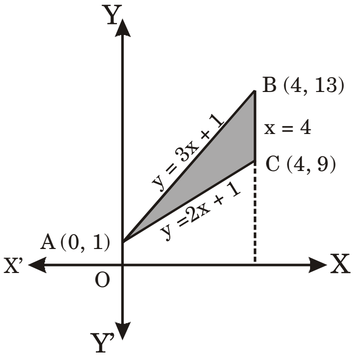
$$-\frac{1}{5}\left[\frac{6^2}{2}+9(6)-\left(\frac{1}{2}+9\right)\right]$$ [1]

$$=\frac{1}{2}\left[\frac{33}{2}-\frac{5}{2}\right]+\frac{1}{3}\left[90-(54)\right]-\frac{1}{5}\left[72-\frac{19}{2}\right]$$

$$=\frac{1}{2}\left(\frac{28}{2}\right)+\frac{1}{3}(36)-\frac{1}{5}\left(\frac{125}{2}\right)$$

$$=7+12-\frac{25}{2}=\frac{13}{2}=6.5\text{squre units}$$ [1]

6.



[1]

The equations of sides of the triangle are:

$y=2x+1$

$y=3x+1$

$x=4$ [1]

On solving these equations, we obtain the vertices of triangle as

$A=(0,1)$

$B=(4,13)$

$C=(4,9)$ [1]

We can observe that,

$$Area(\Delta ACB)=Area(OLBAO)-Area(OLCAO)$$

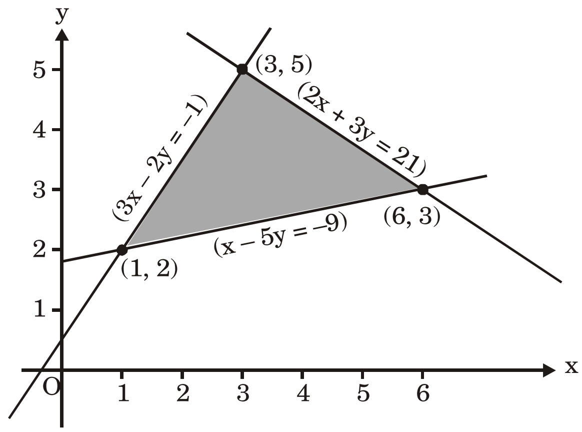
$$=\int_0^4(3x+1)\,dx-\int_0^4(2x+1)\,dx$$ [1]

$$=\left[\frac{3x^2}{2}+x-\frac{2x^2}{2}-x\right]_0^4$$ [1]

$$=\left[\frac{x^2}{2}\right]_0^4$$

= 8 sq. units [1]

7.



Given lines are

$3x-2y+1=0$ ...(i)

$2x+3y-21=0$ ....(ii)

$x-5y+9=0$ ....(iii)

For intersection of (i) and (ii)

Applying (i) × 3 + (ii) × 2, we get

$9x-6y+3+4x+6y-42=0$ [1]

$13x-39=0$

$x=3$

Putting it in (i), we get

$9-2y+1=0$

$\Rightarrow\ 2y=10\Rightarrow y=5$ [1]

Intersection point of (i) and (ii) is (3, 5)

For intersection of (ii) and (iii)

Applying (ii) – (iii) × 2, we get

$2x+3y-21-2x+10y-18=0$

$13y-39=0$

$y=3$ [1]

Putting $y=3$ in (ii), we get

$2x+9-21=0$

$2x-12=0$

$x=6$

Intersection point of (ii) and (iii) is (6, 3)

For intersection of (i) and (iii)

Applying (i) – (iii) × 3, we get

$3x - 2y + 1 - 3x + 15y - 27 = 0$

$13y - 26 = 0 \Rightarrow y = 2$

Putting $y = 2$ in (i), we get

$3x - 4 + 1 = 0$

$x = 1$ [1]

Intersection point of (i) and (iii) is (1, 2) With the help of point of intersection we draw the graph of lines (i), (ii) and (iii)

Shaded region is required region.

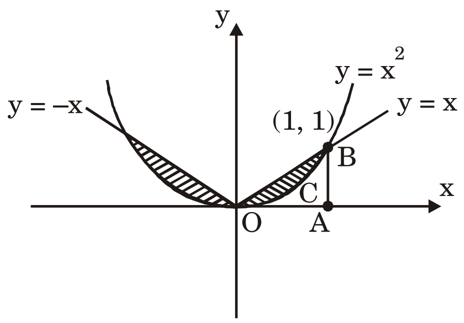
Area of Required region =

$$\int_1^3 \frac{3x+1}{2}\,dx + \int_3^6 \frac{-2x+21}{3} + \int_1^6 \frac{x+9}{5}\,dx$$

After Solving we get,

$$= 10 - \frac{7}{2} = \frac{20-7}{2} = \frac{13}{2}$$ [1]

8.



[1]

Required area

= 2[area of ΔOAB – Area of curve OCBA] [2]

$$A = 2\left[\int_0^1 x\,dx - \int_0^1 x^2\,dx\right]$$ [1]

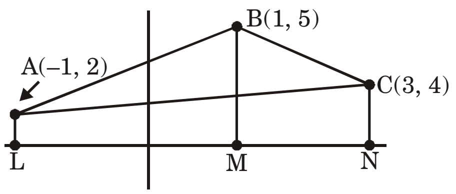
$$A = 2\left[\frac{1}{2} - \frac{1}{3}\right]$$ [1]

$$\Rightarrow A = 2\left[\frac{1}{6}\right] = \frac{1}{3}$$ [1]

9. Let $A = (-1, 2)$

$B = (1, 5)$

$C = (3, 4)$



[1]

We have to find the area of ΔABC

Find eq^n of Line AB

$$y - 5 = \left(\frac{2-5}{-1-1}\right)(x-1)$$

$$y - 5 = \frac{3}{2}(x-1)$$

$2y - 10 = 3x - 3$

$3x - 2y + 7 = 0$

$$y = \frac{3x+7}{2}$$

[1]

Eqn of BC

$$y - 4 = \left(\frac{5-4}{1-3}\right)(x-3)$$

$2y - 8 = -x + 3$

$x + 2y - 11 = 0$

$$y = \frac{11-x}{2}$$ [1]

Eq^n of AC

$$y - 4 = \left(\frac{2-4}{-1-3}\right)(x-3)$$

$\Rightarrow 2y - 8 = x - 3$

$x - 2y + 5 = 0$

$$\int_{-1}^{1}\left(\frac{3x+7}{2}\right)dx + \int_{1}^{3}\left(\frac{11-x}{2}\right)dx + \int_{-1}^{3}\left(\frac{x+5}{2}\right)dx$$ [1]

So, required area

$$\int_{-1}^{1}\left(\frac{3x+7}{2}\right)dx + \int_{1}^{3}\left(\frac{11-x}{2}\right)dx + \int_{-1}^{3}\left(\frac{x+5}{2}\right)dx$$

$$= \frac{1}{2}\left[\frac{3x^2}{2} + 7x\right]_{-1}^{1} + \frac{1}{2}\left[11x - \frac{x^2}{2}\right]_1^3 - \frac{1}{2}\left[\frac{x^2}{2} + 5x\right]_{-1}^{3}$$

[1]

$$= \frac{1}{2}\left[\left(\frac{3}{2}+7\right) - \left(\frac{3}{2}-7\right)\right] + \frac{1}{2}\left[\left(33 - \frac{9}{2}\right) - \left(11 - \frac{1}{2}\right)\right]$$

$$-\frac{1}{2}\left[\left(\frac{9}{2}+15\right) - \left(\frac{1}{2}-5\right)\right]$$

$$= \frac{1}{2}[14 + 22 - 4 - 24] = 4 \text{ sq. units.}$$ [1]

10. $y = 2 + x$ ...(1)

$y = 2 - x$ ...(2)

$x = 2$ ...(3)

y_1 is the value of y from (1)

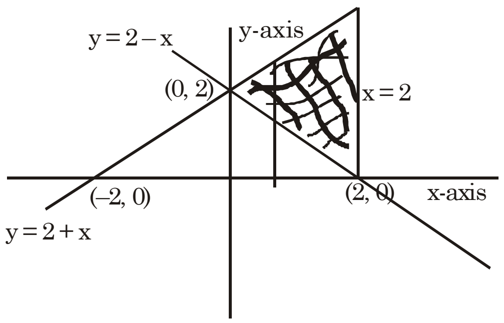
And y_2 is the value of y from (2)

Required area $= \int_0^2 (y_1 - y_2)\, dx$ [1]

$= \int_0^2 \{(2+x)-(2-x)\}\, dx$ [1]

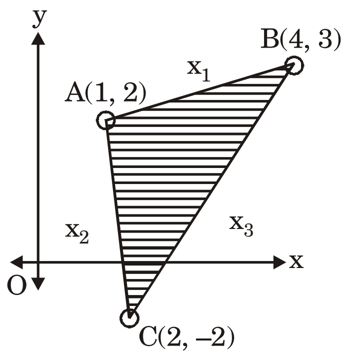
$= 2\int_0^2 x dx = 2\left[\frac{x^2}{2}\right]_0^2$ [1]

$= 4\ sq.units$ [1]



[2]

11.



[1]

Equation of line AB:

$y-2=\left(\frac{3-2}{4-1}\right)(x-1)$

$y-2=\frac{1}{3}(x-1)$

$\Rightarrow x = 3y-5$ (a) [1]

Equation of line AC:

$y-2=\left(\frac{-2-2}{2-1}\right)(x-1)$

$\Rightarrow y-2=-4x+4$

$\Rightarrow x=\frac{6-y}{4}$ (b) [1]

Equation of line BC:

$y+2=\left(\frac{3+2}{4-2}\right)(x-2)$

$2y+4=5x-10$

$x=\frac{2y+14}{5}$ (c) [1]

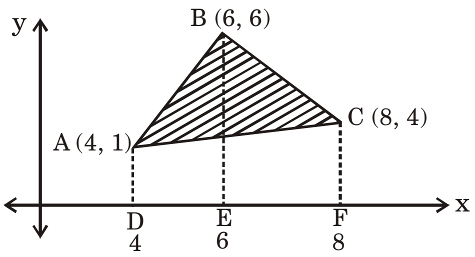
Area of $\triangle ABC$

$\Rightarrow \int_{-2}^{2}(x_3-x_2)\,dy+\int_{2}^{3}(x_3-x_1)\,dy$ [1]

$\Rightarrow \int_{-2}^{2}\left(\frac{2y+14}{5}-\frac{6-y}{4}\right)dy+\int_{2}^{3}\left(\frac{2y+14}{5}-(3y-5)\right)dy$

$\Rightarrow \frac{13}{2}$sq.units [1]

12.



[1]

Equation of AB is $y-1=\frac{6-1}{6-4}(x-4)$

$\Rightarrow 2y - 2 = 5x - 20$

$\Rightarrow y=\frac{5x}{2}-9$ [1]

Equation of BC is

$\Rightarrow y-6=\frac{4-6}{8-6}(x-6)$

$\Rightarrow y = -x+12$ [1]

Equation of AC is

$\Rightarrow y-1=\frac{4-1}{8-4}(x-4)$

$\Rightarrow 4y-4=3x-12$

$\Rightarrow y=\frac{3x}{4}-2$ [1]

Area of $\triangle$ABC = Area ABED + Area BEFC – Area ADFC

$=\int_4^6\left(\frac{5x}{2}-9\right)dx+\int_6^8(-x+12)\,dx-\int_4^8\left(\frac{3x}{4}-2\right)dx$ [1]

$$= \left|\left(\frac{5x^2}{4} - 9x\right)_4^6\right| + \left|\left(\frac{-x^2}{2} + 12x\right)_6^8\right| - \left|\left(\frac{x^2}{8} - 2x\right)_4^8\right|$$

$= 7$ sq units [1]

13. Parabola $4y = 3x^2$ ...(1)

line $3x - 2y + 12 = 0$ ...(2)

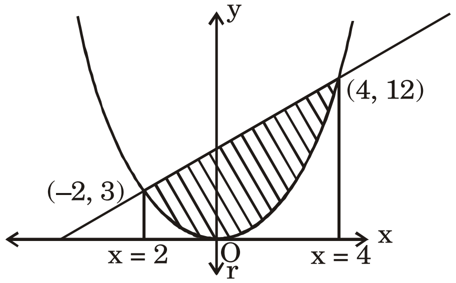
from (2) $y = \frac{3x+12}{2}$ putting this value of y in eq. (1) we get

$6x + 24 = 3x^2$ [1]

$\Rightarrow x = 4, -2$ when $x = 4$ then $y = 12$ [1]

$x = -2$ then $y = 3$

$$= \int_{-2}^{4} (y \text{ of line})\,dx = \int_{-2}^{4} (y \text{ of parabola})\,dx \quad [1]$$



[1]

$$= \int_{-2}^{4} \left(\frac{3x+12}{2} - \frac{3x^2}{4}\right) dx$$

$$= \frac{3}{4}\int_{-2}^{4} \left(8 + 2x - x^2\right) dx \quad [1]$$

$$= \frac{3}{4}\left[8x + x^2 - \frac{x^3}{3}\right]_{-2}^{4}$$

$= 27$ sq units [1]

14. The area of the region bounded by the circle $x^2 + y^2 = 32$, line $y = x$ and the x-axis is the area OAB.

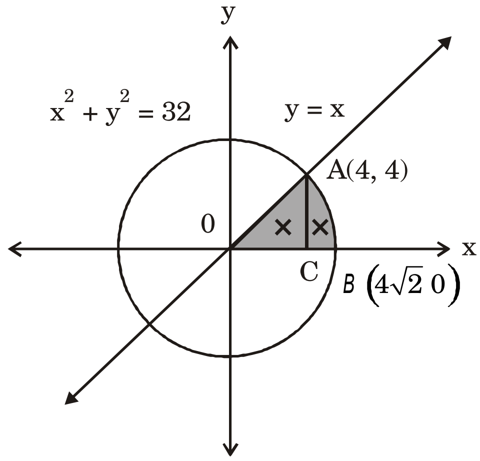
The point of intersection of the line and the circle in the first quadrant is $(4, 4)$. [1]

Line $y = x$

Curve

$x^2 + y^2 = 32$

$y = \sqrt{32 - x^2}$ [1]



[1]

Area OAB = Area ΔOCA + Area ACB

Area OAB = Area under line + Area under curve.

$$= \int_0^4 y\,dx + \int_4^{4\sqrt{2}} y\,dx \quad [1]$$

$$= \int_0^4 x\,dx + \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\,dx$$

Using $\int \sqrt{a^2 - x^2} = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

$$= \left[\frac{x^2}{2}\right]_0^4 + \left[\frac{x}{2}\sqrt{32 - x^2} + \frac{32}{2}\sin^{-1}\frac{x}{\sqrt{32}}\right]_4^{4\sqrt{2}}$$

$$= \frac{1}{2}\left[x^2\right]_0^4 + \frac{1}{2}\left[x\sqrt{32 - x^2} + 32\sin^{-1}\frac{x}{\sqrt{32}}\right]_4^{4\sqrt{2}}$$

$$= \frac{1}{2}\left[4^2 - 0^2\right] + \frac{1}{2}\left[\begin{array}{l} 4\sqrt{32 - 32} + 32\sin^{-1}\frac{4\sqrt{2}}{\sqrt{32}} \\ -4\sqrt{32 - 4^2} - 32\sin^{-1}\frac{4}{\sqrt{32}} \end{array}\right] \quad [1]$$

$$= 8 + \frac{1}{2}\left[0 + 32\left(\frac{\pi}{2}\right) - 16 - 32\left(\frac{\pi}{4}\right)\right]$$

$= 8 - 8 + 8\pi - 4\pi$

$= 4\pi$ square units

Therefore, area enclosed by x-axis, the line $y = x$, and the circle $x^2 + y^2 = 32$ in the first quadrant is 4π square units . [1]

15. We have, $x^2 + y^2 = 4$

Diffrentiating both sides w.r.t. x

$$2x + 2y\frac{dy}{dx} = 0$$ [1]

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dy}{dx} at\left(1, \sqrt{3}\right) = -\frac{1}{\sqrt{3}} = \text{slope of tangent } (m_1)$$ [1]

Slope of normal, $m_2 = +\sqrt{3}$

Equation of tangent at $\left(1, \sqrt{3}\right)$ is

$$y - y_1 = m_1\left(x - x_1\right)$$

$$y - \sqrt{3} = -\frac{1}{\sqrt{3}}\left(x - 1\right)$$

$$\sqrt{3}y - 3 = -x + 1$$

$$x + \sqrt{3}y = 4$$

$$y = \frac{4 - x}{\sqrt{3}}$$ [1]

X	4	1
Y	0	$\sqrt{3}$

[1]

Equation of normal at $\left(1, \sqrt{3}\right)$

$$y - y_1 = m_2\left(x - x_1\right)$$

$$y - \sqrt{3} = \sqrt{3}\left(x - 1\right)$$

$$y - \sqrt{3} = \sqrt{3}x - \sqrt{3}$$

$$-\sqrt{3} + \sqrt{3} = \sqrt{3}x - y$$

$$\sqrt{3}x - y = 0$$

$$-y = -\sqrt{3}x$$

$$y = \sqrt{3}x$$

The area of $\Delta OAB = \int_0^1 \sqrt{3}x\,dx + \int_1^4 \left(\frac{4 - x}{\sqrt{3}}\right)dx$

$$= \frac{\sqrt{3}}{2}\left[x^2\right]_0^1 + \frac{1}{\sqrt{3}}\left[4x - \frac{x^2}{2}\right]_1^4$$ [1]

$$= \frac{\sqrt{3}}{2}\left(1^2 - 0\right) + \frac{1}{\sqrt{3}}\left[\left(4(4) - \frac{16}{2}\right) - \left(4(1) - \frac{1^2}{2}\right)\right]$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3}}\left(\frac{9}{2}\right) = \frac{6\sqrt{3}}{2} = 2\sqrt{3} \text{ sq. units.}$$ [1]

Smart Notes ....

Smart Notes ....

CHAPTER 9

Differential Equations

Chapter Analysis of Previous Three Years' Board Exams

Number of Questions Asked in Exams

Topics	2018		2017		2016	
	Delhi	**All India**	**Delhi**	**All India**	**Delhi**	**All India**
Forming Differential Equation	2 marks	2 marks				4 marks
General Solution			4 marks		4 marks	
Particular Solution	4 marks	4 marks		6 marks	4 marks	
Solve Differential Equation			6 marks	4 marks		4 marks

[TOPIC 1] Formation of Differential Equations

Summary

- **Differential Equation:** Differential equation is an equation which involves derivative of the dependent variable with respect to independent variable. Here $x\frac{dy}{dx}+y=0$ is the example of differential equation.
- General notations for derivatives are:

$$\frac{dy}{dx}=y', \frac{d^2y}{dx^2}=y'', \frac{d^3y}{dx^3}=y'''$$

or $\frac{d^ny}{dx^n}=y_n$

- **Order of a differential equation:** The order of the highest order derivative in any given differential equation is called the order of the differential equation.

Example: $\frac{dy}{dx}-\sin x=0$ has the order 2.

- **Degree of a differential equation:** If a differential equation involves a polynomial equation in its derivative, then its degree can be defined as the highest power of the highest order derivative in it.

Example: $\frac{dy}{dx}-\sin x=0$ has the degree 1.

Degree (if defined) and order are always positive.

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

1 Mark Questions

1. Find the differential equation representing the curve $y = cx + c^2$.

[ALL INDIA 2015]

2. Write the differential equations representing the family of curves $y = mx$, where m is an arbitrary constant.

[ALL INDIA 2013]

3. Write the degree of the differential equation

$$x^3\left(\frac{d^2y}{dx^2}\right)^2+x\left(\frac{dy}{dx}\right)^4=0.$$

[DELHI 2013]

4. Find the differential equation representing the family of curves $\upsilon=\frac{A}{r}+B,$ where A and B are arbitrary constants

[DELHI 2015]

2 Marks Question

5. Find the differential equation representing the family of curves $y=ae^{bx+5}$, where a and b are arbitrary constants.

[DELHI 2018]

4 Marks Questions

6. Find the differential equation for all the straight lines, which are at a unit distance from the origin.

[ALL INDIA 2015]

Solutions

1. $\frac{dy}{dx} = c$

$$y = x\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^3$$ [1]

2. $y = mx$ ...(1)

differentiating with respect to x, we get

$$\frac{dy}{dx} = m$$

∴ Differential equation of the curve.

$$y = \frac{xdy}{dx}$$ [1]

3. Given : $x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0.$

Degree =2 [1]

4. $\upsilon = \frac{A}{r} + B,$

Diffrentiating both sides w.r.t, r, we have

$$\frac{dv}{dr} = A\left(\frac{-1}{r^2}\right) \Rightarrow r^2\left(\frac{dv}{dr}\right) = -A$$ [½]

Again differentiating both sides w.r.t. r, we have

$$r^2.\frac{d^2v}{dr^2} + \frac{dv}{dr}(2r) = 0$$

Dividing both sides by r

$$r.\frac{d^2v}{dr^2} + 2\frac{dv}{dr} = 0$$ [½]

5. $y = ae^{bx+5}$

Differentiating both sides with respect to x, we get:

$$y' = abe^{bx+5}$$

From 1 we get

$$y' = by$$

$$\frac{y'}{y} = b$$ [1]

Again, differentiating both sides with respect to x, we get:

$$\frac{yy'' - y' \times y'}{y^2} = 0$$

$$\Rightarrow yy'' - (y')^2 = 0$$

$$\Rightarrow yy'' = (y')^2$$

This is the required differential equation of the given curve. [1]

6. Let the equation of line be $y = mx + c$

the line is at unit distance from the origin

i.e. $\left|\frac{0+c}{\sqrt{1+m^2}}\right| = 1$ [1]

$$\Rightarrow c = \sqrt{1+m^2}$$

$$\therefore y = mx + \sqrt{1+m^2}$$ [1]

$$\frac{dy}{dx} = m$$ [1]

$$\therefore y = x\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$ [1]

[TOPIC 2] Solution of Different Types of Differential Equations

- **Solution of a differential equation:** Solution of a differential equation is a function satisfying that differential equation.
  - ➢ **General Solution:** The solution having the arbitrary constants (equal to the order of the differential equation) is called a general or primitive solution of the differential equation.
  - ➢ **Particular Solution:** The solution which does not have the arbitrary constants is called a particular solution. It is acquired by substituting the particular values in the arbitrary constants.
- If the general solution of any differential equation is given, then the function is to be differentiated successively (as many times as the total number of arbitrary constants) in order to form the differential equation and then the arbitrary constants are eliminated.
- A differential equation which can be separated completely such that the terms which contains x can be written with dx and that of containing y with dy, can be solved with the help of variable separable method. The solution of such equations are of the form $\int f(x)dx = \int g(x)dx + C$, where C is an arbitrary constant.
- **Homogeneous Differential Equation:** A differential equation of the form $\frac{dy}{dx} = f(x,y)$ is called as homogeneous differential equation if f (x, y) is a homogeneous function of degree 0.
- To solve a homogeneous equation of the form $\frac{dy}{dx} = f(x,y) = g\left(\frac{y}{x}\right)$, substitutions are made as $\frac{y}{x} = v$ or y = vx and then general solution is found out by solving $\frac{dy}{dx} = v + x\frac{dv}{dx}$.
- To solve a homogeneous equation of the form $\frac{dx}{dy} = f(x,y)$, substitutions are made as $\frac{x}{y} = v$ or x = vy and then general solution is found out by writing $\frac{dx}{dy} = v + y\frac{dv}{dy}$.
- **Linear Differential Equation:** A differential equation which can be expressed as $\frac{dy}{dx} + Py = Q$, where P and Q are the constants or functions of x only, is called a linear differential equations of first order.
- To solve a linear differential equation, it is first written as $\frac{dy}{dx} + Py = Q$, then the integrating factor is found as $I.F. = e^{\int P dx}$. After that, the solution is given by $y(I.F.) = \int (Q \times I.F.)dx + C$.
- If the linear differential equation is of the form $\frac{dx}{dy} + Px = Q$ (P and Q are constants or functions of y only), then $I.F. = e^{\int P dy}$ and the solution is given by $x(I.F.) = \int (Q \times I.F.)dy + C$.

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 2

1 Mark Questions

1. Write the integrating factor of the following differential equation:

$$\left(1+y^2\right)dx-\left(\tan^{-1}y-x\right)dy=0$$

[ALL INDIA 2015]

2. Find the integrating factor of the differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dx}{dy}=1$

[DELHI 2015]

4 Marks Questions

3. Prove that $x^2 - y^2 = c(x^2 + y^2)^2$ is the general solution of the differential equation $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$, where C is a parameter.

[DELHI 2013]

4. Find the particular solution of the differential equation:

$\left(1-y^2\right)(1+\log x)\,dx+2xy\,dy=0$ given that $y = 0$, when $x = 1$

[DELHI 2016]

5. Find the general solution of the following differential equation:

$$\left(1+y^2\right)+\left(x-e^{\tan^{-1}y}\right)\frac{dy}{dx}=0$$

[DELHI 2016]

6. Find the particular solution of the differential equation $e^x\sqrt{1-y^2}\,dx+\frac{y}{x}dy=0$, given that $y = 1$ when $x = 0$.

[DELHI 2014]

7. Solve the following differential equation:

$$\left(x^2-1\right)\frac{dy}{dx}+2xy=\frac{2}{x^2-1}.$$

[DELHI 2014]

8. Solve the following differential equation:

$$2x^2\frac{dy}{dx}-2xy+y^2=0$$

[DELHI 2012]

9. Find the particular solution of the following differential equation:

$\frac{dy}{dx}=1+x^2+y^2+x^2y^2$, given that $y=1$ when $x=0$

[DELHI 2012]

10. Solve the following differential equation:

$$e^x\tan y\,dx+\left(1-e^x\right)\sec^2 y\,dy=0$$

[DELHI 2011]

11. Solve the following differential :

$$\cos^2 x\frac{dy}{dx}+y=\tan x.$$

[DELHI 2011]

12. Find the particular solution of the differential equation:

$$2xy+y^2-2x^2\frac{dy}{dx}=0;\ y=2,\text{ when } x=1$$

[ALL INDIA 2011]

13. Solve the following differential equation:

$\frac{dy}{dx}+y\cot x=4x\,cosecx$, given that $y=0$ when $x=\frac{\pi}{2}$

[All India 2011]

14. If $y=Pe^{ax}+Qe^{bx}$, show that

$$\frac{d^2y}{dx^2}-(a+b)\frac{dy}{dx}+aby=0$$

[ALL INDIA 2014]

15. Show that the differential equation $\frac{2xydy}{dx}=x^2+3y^2$ is homogeneous and solve it.

[ALL INDIA 2015]

16. Find the particular solution of the differential equation $2ye^{\frac{x}{y}}dx+\left(y-2xe^{\frac{x}{y}}\right)dy=0$

Given $x = 0$ that when $y = 1$

[ALL INDIA 2016]

17. Find the particular solution of differential equation: $\frac{dy}{dx}=-\frac{x+ycosx}{1+sinx}$ given that $y=1$ when $x=0$

[ALL INDIA 2016]

18. Solve the differential equation

$$\left(\tan^{-1}x-y\right)dx=\left(1+x^2\right)dy$$

[ALL INDIA 2017]

19. Find the particular solution of the differential equation $e^x\tan ydx+\left(2-e^x\right)\sec^2 ydy=0$, given that $y=\frac{\pi}{4}$ when $x=0$

[DELHI 2018]

20. Find the particular solution of the differential equation $\frac{dy}{dx}+2y\tan x=\sin x$, given that $y=0$ when $x=\frac{\pi}{3}$.

DELHI 2018]

21. Solve the differential equation

$$x\frac{dy}{dx}+y=x\cos x+\sin x,$$

given that $y=1$ when $x=\frac{\pi}{2}$

[DELHI 2017]

22. Find the particular solution of the differential equation $\frac{dy}{dx}=1+x+y+xy$, given that $y=0$ when $x=1$.

[ALL INDIA 2014]

23. Show that the differential equation $2ye^{x/y}+(y-2xe^{x/y})dy=0$ is homogeneous .Find the particular solution of this different equation, given that

$x=0$ when $y=1$

[2013]

6 Marks Questions

24. Find the particular solution of the differential equation $\left(\tan^{-1}y-x\right)dy=\left(1+y^2\right)dx$, given that when $x=0$, $y=0$.

[ALL INDIA 2013]

25. Find the particular solution of the differential equation $(x-y)\frac{dy}{dx}=(x+2y)$ given that $y=0$ when $x=1$.

[ALL INDIA 2017]

26. Find the particular solution of the differential equation: $\frac{dy}{dx}=\frac{xy}{x^2+y^2}$ given that $y=1$, when $x=0$

[DELHI 2015]

Solutions

1. $\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2}$

This is of the form dx + px = Q

I.F. = $e^{\int pdx}$

I.F. = $e^{\tan^{-1}y}$ [1]

2. $\left(\frac{e-2\sqrt{x}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dx}{dy}=1$

$$\left(\frac{e-2\sqrt{x}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)=\frac{dy}{dx}$$

$$\frac{dy}{dx}+\frac{y}{\sqrt{x}}=\frac{e-2\sqrt{x}}{\sqrt{x}}$$

This is of the form $\frac{dy}{dx}+p'y=Q$

Here, $P'=\frac{1}{\sqrt{x}}=x^{-\frac{1}{2}}$

Integrating factor

$=e^{\int p.dx}=e^{\int x^{-1/2}}=e^{2x^{1/2}}$ or $e^{2\sqrt{x}}$ [1]

3. Given $x^2-y^2=c\left(x^2+y^2\right)^2$

$\frac{x^2-y^2}{\left(x^2+y^2\right)^2}=c$ (1) [1]

Differentiating both sides w.r.t x we get

$$2x-2y\frac{dy}{dx}=2\left(x^2+y^2\right)\left(2x+2y\frac{dy}{dx}\right)$$

From (1) we get

$$\Rightarrow 2\left(x - y\frac{dy}{dx}\right) \quad [1]$$

$$= 2\left(\frac{x^2 - y^2}{\left(x^2 + y^2\right)^2}\right)\left(x^2 + y^2\right)\left(2x + 2y\frac{dy}{dx}\right)$$

$$\Rightarrow \left(x - y\frac{dy}{dx}\right)\left(x^2 + y^2\right)$$

$$= \left(x^2 - y^2\right)\left(2x + 2y\frac{dy}{dx}\right)$$

$$\Rightarrow \left(x - y\frac{dy}{dx}\right)\left(x^2 + y^2\right)$$

$$= \left(x^2 - y^2\right)\left(2x + 2y\frac{dy}{dx}\right)$$

$$\Rightarrow -x^2 y\frac{dy}{dx} - y^3\frac{dy}{dx} - 2x^2 y\frac{dy}{dx} + 2y^3\frac{dy}{dx} \quad [1]$$

$$= 2x^3 - 2xy^2 - x^3 - xy^2$$

$$\Rightarrow \frac{dy}{dx}\left(-x^2 y - y^3 - 2x^2 y + 2y^3\right)$$

$$= 2x^3 - 2xy^2 - x^3 - xy^2$$

$$\frac{dy}{dx}\left(-3x^2 y + y^3\right) = \left(x^3 - 3xy^2\right)$$

$$\left(y^3 - 3x^2 y\right)dy = \left(x^3 - 3xy^2\right)dx$$

Thus, $x^2 - y^2 = c\left(x^2 + y^2\right)^2$ is the solution of given differential equation. [1]

4. $(1 - y)^2(1 + \log x)\,dx + 2xydy = 0$

$$2xydy = -\left(1 - y^2\right)(1 + \log x)\,dx$$

$$\frac{2ydy}{-\left(1 - y^2\right)} = \frac{1 + \log x}{x}dx \quad [1]$$

Integrating on both sides, we get

$$\int\frac{2ydy}{y^2 - 1} = \int\left(\frac{1 + \log x}{x}\right)dx$$

$$\begin{bmatrix}\text{let } p = y^2 - 1, & \text{let } q = 1 + \log x\\ dp = 2ydy & dq = \frac{1}{x}dx\end{bmatrix} \quad [1]$$

$$\int\frac{dp}{p} = \int qdq$$

$$\log|p| = \frac{q^2}{2} + C$$

$$\log\left|y^2 - 1\right| = \frac{1}{2}(1 + \log x)^2 + C$$

Now, $x = 1, y = 0 \Rightarrow c = -\frac{1}{2}$

Using $c = -\frac{1}{2}$, we get

$$\log\left|y^2 - 1\right| = \frac{1}{2}(1 + \log x)^2 - \frac{1}{2}$$

$$2\log\left|y^2 - 1\right| = (1 + \log x)^2 - 1$$

$$\therefore (1 + \log x)^2 - 2\log\left|y^2 - 1\right| = 1$$

Is the particular solution of the differential equation [1]

5. $\left(1 + y^2\right) + \left(x - e^{\tan^{-1}y}\right)\frac{dy}{dx} = 0$

$$\left(x - e^{\tan^{-1}y}\right)\frac{dy}{dx} = -\left(1 + y^2\right)$$

$$\frac{dy}{dx} = \frac{-\left(1 + y^2\right)}{x - e^{\tan^{-1}y}} \quad [1]$$

Taking reciprocal on both sides, we have

$$\frac{dx}{dy} = \frac{x - e^{\tan^{-1}y}}{-\left(1 + y^2\right)}$$

$$\frac{dx}{dy} = \frac{-x}{1 + y^2} + \frac{e^{\tan^{-1}y}}{1 + y^2}$$

$$\frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{e^{\tan^{-1}y}}{1 + y^2}$$

Comparing with $\frac{dx}{dy} + \mathrm{P}x = Q$

$\therefore P = \frac{1}{1+y^2}, Q = \frac{e^{\tan^{-1}y}}{1+y^2}$

I.F. $= e^{\int P dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

Hence, the solution is $x(\text{I.F.}) = \int Q(\text{I.F})\,dy$ [1]

$x\left(e^{\tan^{-1}y}\right) = \int \frac{e^{\tan^{-1}y}}{1+y^2} e^{\tan^{-1}y} dy$

Let $t = e^{\tan^{-1}y}$

$dt = e^{\tan^{-1}y}.\frac{1}{1+y^2}dy$ [1]

$x\left(e^{\tan^{-1}y}\right) = \int t\,dt$

$x\left(e^{\tan^{-1}y}\right) = \frac{t^2}{2} + c$

$x\left(e^{\tan^{-1}y}\right) = \frac{\left(e^{\tan^{-1}}\right)^2}{2} + c$

$x = \frac{\left(e^{\tan^{-1}y}\right)^2}{2\left(e^{\tan^{-1}y}\right)} + \frac{c}{\left(e^{\tan^{-1}y}\right)}$

$x = \frac{1}{2}\left(e^{\tan^{-1}y}\right) + c.e^{-\tan^{-1}y}$ [1]

6. $e^x\sqrt{1-y^2}\,dx + \frac{y}{x}dy = 0$

$\frac{y}{x}dy = -e^x\sqrt{1-y^2}\,dx$

$\frac{-y}{\sqrt{1-y^2}}dy = xe^x dx$

$\int \frac{-y}{\sqrt{1-y^2}}dy = \int xe^x dx$ [1]

Using by parts method.

Let $p = 1-y^2$, $dp = -2y\,dy$ $\frac{dp}{2} = -y.dy$ [1]

$\frac{1}{2}\int \frac{dp}{\sqrt{p}} = \left[xe^x - \int 1.e^x dx\right]$

$\frac{1}{2}\int p^{-\frac{1}{2}}dp = xe^x - e^x + c$

$\frac{1}{2}.\frac{2}{1}p^{\frac{1}{2}} = e^x(x-1) + c$ [1]

$\sqrt{1-y^2} = e^x(x-1) + c$ (i)

$\because y = 1, x = 0$

$\sqrt{1-1} = e^0(0-1) + c_1$

$0 = 1(-1) + c$

$\Rightarrow c = 1$

Putting the value of c in (i),

$\sqrt{1-y^2} = e^x(x-1) + 1$, which is the particular solution. [1]

7. $(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$

Dividing both sides by (x^2-1)

$\frac{dy}{dx} + \frac{2xy}{(x^2-1)} = \frac{2}{(x^2-1)^2}$ [1]

Comparing with $\frac{dy}{dx} + Py = Q$

$P = \frac{2x}{x^2-1}, Q = \frac{2}{\left(x^2-1\right)^2}$

Let $t = x^2-1, dt = 2x\,dx$

I.F $= e^{\int P dx} = e^{\int \frac{2x}{x^2-1}dx}$ [1]

$= e^{\int \frac{dt}{t}} = e^{\log|t|}$

$= \left[e^{\log|x|} = x\right] = |t| = x^2 - 1$

Hence, the solution is

$$y(\text{I.F.}) = \int Q(\text{I.F}).\text{dx}$$

$$y(x^2 - 1) = \int \frac{2}{\left(x^2-1\right)^2}\left(x^2-1\right)dx$$ [1]

$$y(x^2 - 1) = \int \frac{2}{\left(x^2-1\right)}dx$$

$$y(x^2 - 1) = 2.\frac{1}{2.1}\log\left|\frac{x-1}{x+1}\right| + c$$

$$\left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{(x^2-1)}\log\left|\frac{x-1}{x+1}\right| + \frac{c}{(x^2-1)}$$ [1]

8. Given $2x^2\frac{dy}{dx} - 2xy + y^2 = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy - y^2}{2x^2} \qquad \text{(i)}$$ [1]

Let $y = vx$

$$\Rightarrow \frac{y}{x} = v \qquad \text{(ii)}$$

$$\frac{dy}{dx} = v.1 + \text{x}.\frac{dv}{dx}$$

Replacing (ii) in (i)

$$2x^2\frac{dy}{dx} = 2xy - y^2$$

$$\frac{dy}{dx} = \frac{2xy}{2x^2} - \frac{y^2}{2x^2}$$ [1]

$$\frac{dy}{dx} = \frac{y}{x} - \frac{1}{2}\left(\frac{y}{x}\right)^2$$

$$v + x\frac{dv}{dx} = v - \frac{1}{2}v^2$$ [1]

$$-2\frac{dv}{v^2} = \frac{dx}{x}$$

$$-2\int v^{-2}\ dv = \int \frac{dx}{x}$$

$$\frac{-2v^{-1}}{-1} = \log|x| + C$$

$$\frac{2}{v} = \log|x| + C$$

$$2x = y\log|x| + Cy$$ [1]

9. Given : $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$

$$= 1\left(1+x^2\right) + y^2\left(1+x^2\right)$$ [1]

$$= \left(1+x^2\right)\left(1+y^2\right)$$

$$\frac{dy}{1+y^2} = \left(1+x^2\right)dx$$

Integrating both sides

$$\int \frac{dy}{1+y^2} = \int\left(1+x^2\right)dx$$ [1]

$$\tan^{-1}(y) = x + \frac{x^3}{3} + C$$

$$\tan^{-1}(1) = 0 + 0 + C$$ [1]

$$\frac{\pi}{4} = C$$

Putting the value of C in (i), we have

$$\tan^{-1}(y) = x + \frac{x^3}{3} + \frac{\pi}{4}$$ [1]

10. Given: $e^x\tan y\,dx + \left(1-e^x\right)\sec^2 y\,dy = 0$

using $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$

$$\left(1-e^x\right)\sec^2 y\,dy = -e^x\tan y\,dx$$

$$\frac{\sec^2 y}{\tan y}dy = \frac{-e^x}{1-e^x}dx$$ [1]

Integrating both sides ,

$$\int \frac{\sec^2 y}{\tan y}dy = \int \frac{-e^x}{1-e^x}dx$$ [1]

Let p = tany

$dp = \sec^2 y\ dy$

q = $1 - e^x$

$dq = -e^x dx$ [1]

$$\int \frac{d}{p} = \int \frac{1}{q} dq$$

$$\log|p| = \log|q| + \log c$$

$$\log|\tan y| = \log\left|1 - e^x\right| + \log c$$

$$\log|\tan y| = \log\left|c\left(1 - e^x\right)\right|$$

$$\tan y = c\left(1 - e^x\right)$$ [1]

11. Given differential equation is

$$\cos^2 x\frac{dy}{dx} + y = \tan x.$$

Dividing both sides by $\cos^2 x$, we have

$$\frac{dy}{dx} + \frac{y}{\cos^2 x} = \frac{\tan x}{\cos^2 x}$$ [1]

$$\frac{dy}{dx} + y\sec^2 x = \tan x.\sec^2 x$$

Comparing $\frac{dy}{dx} + py = Q$ = we have

$$p = \sec^2 x, Q = \tan x\sec^2 x$$ [1]

Integrating factor,

$$\text{IF} = e^{\int p dx} = e^{\int \sec^2 x dx} = e^{\tan x}$$

Solution of equation is y (IF) = $\int Q.(\text{IF})\, dx$

$$y.e^{\tan x} = \int \tan x.\sec^2 x.e^{\tan x} dx$$

Let $t = \tan x$

$$\frac{dt}{dx} = \sec^2 x$$

$$dt = \sec^2 x\, dx$$

$$y.e^{\tan x} = \int t.e^t dt$$ [1]

Using by parts

$$y.e^{\tan x} = te^t - \int 1.e^t dt$$

$$y.e^{\tan x} = te^t - e^t + C = e^t.(t-1) + C$$

$$Y = \frac{e^{\tan x}(\tan x - 1)}{e^{\tan x}} + \frac{C}{e^{\tan x}}$$

$$y = \tan x - 1 + Ce^{-\tan x}$$ [1]

12. $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{\left(2xy + y^2\right)}{2x^2} \qquad (1)$$

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$ [1]

Put value of y & $\frac{dy}{dx}$ in equation (1)

$$v + x\frac{dv}{dx} = \frac{2vx^2 + v^2x^2}{2x^2}$$

$$x\frac{dv}{dx} = \frac{2v + v^2}{2} - v$$

$$x\frac{dv}{dx} = \frac{2v + v^2 - 2v}{2}$$ [1]

$$x\frac{dv}{dx} = \frac{v^2}{2}$$

$$\frac{2}{v^2} dv = \frac{dx}{x}$$

$$2v^{-2} dv = \frac{dx}{x}$$

Integrating both sides

$$\int 2v^{-2} dv = \int \frac{dx}{x}$$

$$-2v^{-1} = \log x + C$$

$$\frac{-2x}{y} = \log x + C$$

When $y = 2$ and $x = 1$

$$-\frac{2}{2} = \log 1 + C$$

$C = -1$ [1]

Now the equation becomes as

$$\frac{-2x}{y} = \log x - 1$$

$$-2x = y\log x - y$$

$$y = y\log x + 2x$$

$$y = \frac{2x}{1 - \log x}$$

This is the required solution of the given differential equation. [1]

13. $\frac{dy}{dx} + y\cot x = 4x\,\mathrm{cosec}\,x$

$\frac{dy}{dx} + \cot x.y = 4x\,\mathrm{cosec}\,x$

$\frac{dy}{dx} + Py = Q$ [1]

Where $P = \cot x$, $Q = 4\ x\ \mathrm{cosec}\ x$,

I.F. $= e^{\int P dx} = e^{\int \cot x dx} = e^{\log \sin x} = \sin x$

Now multiplying the equation by I.F.

$y\sin x = \int \sin x \times 4\frac{x}{\sin x}$ [1]

$y\sin x = \frac{24x^2}{2} + C$

$y\sin x = 2x^2 + C$

$y = 0,\ x = \frac{\pi}{2}$ [1]

$C = \frac{2\pi^2}{4}$

$y\sin x = 2x^2 - \frac{\pi^2}{2}$

This is required particular solution of the given differential equation. [1]

14. $y = Pe^{ax} + Qe^{bx}$ ...(1)

$\frac{dy}{dx} = aPe^{ax} + bQe^{bx}$ ...(2)

$\frac{d^2y}{dx^2} = a^2Pe^{ax} + b^2Qe^{bx}$ ...(3) [1]

multiplying(1) by ab

we get, $aby = abPe^{ax} + abQe^{bx}$ ...(4)

multiplying (2) by (a + b)

we get,

$(a+b)\frac{dy}{dx} = (a+b)\left(aPe^{ax} + bQe^{bx}\right)$ [1]

$= (a^2Pe^{ax} + b^2Qe^{bx}) + \left(abPe^{ax} + abQe^{bx}\right)$

Or

$\left(a^2Pe^{ax} + b^2Qe^{bx}\right) - (a+b)\frac{dy}{dx} + (aPe^{ax} + bQe^{bx})$

[1]

Or

$\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = 0$ [1]

15. $\frac{dy}{dx} = \frac{(x^2+3y^2)}{2xy} = \frac{1+3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}$ ...(1) [1]

Differential equation is homogeneous

Put $y = vx$ ...(2)

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting (1) and (2)

$\therefore\ v + x\frac{dv}{dx} = \frac{1+3v^2}{2v}$ [1]

$x\frac{dv}{dx} = \frac{1+v^2}{2v}$

$\int \frac{2v}{1+v^2}dv = \int \frac{dx}{x}$

$\Rightarrow \log\left|1+v^2\right| = \log|x| + \log c$ [1]

$1 + v^2 = cx$

$1 + \left(\frac{y}{x}\right)^2 = cx$ or $x^2 + y^2 = cx^3$ [1]

16. $2ye^{\frac{x}{y}}dx + \left(y - 2xe^{\frac{x}{y}}\right)dy = 0$

$\Rightarrow 2e^{\frac{x}{y}} = -\left(1 - \frac{2x}{y}e^{\frac{x}{y}}\right)\frac{dy}{dx}$

Put $\frac{x}{y} = t$ [1]

$x = yt$

$1 = \frac{dy}{dx}t + y\frac{dt}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{t}\left(1 - y\frac{dt}{dx}\right)$

$$\Rightarrow 2e^t = -\left(1-2te^t\right).\frac{1}{t}\left(1-\frac{ydt}{dx}\right)$$

$$\Rightarrow 2e^t = -\left(1-2te^t\right).\frac{1}{t}\left(1-y\frac{dt}{dx}\right) \qquad [1]$$

$$\Rightarrow 2te^t = \left(2te^t-1\right).\left(1-y\frac{dt}{dx}\right)$$

$$= 2te^t - 1 - y.2te^t\frac{dt}{dx} + \frac{ydt}{dx}$$

$$\Rightarrow 1 = y\left(1-2te^t\right).\frac{dt}{dx}$$

$$= \frac{x}{t}\left(1-2te^t\right).\frac{dt}{dx}$$

$$\Rightarrow \int\frac{dx}{x} = \int\left(\frac{1}{t}-2e^t\right)dt$$

$$\Rightarrow \ln x = \ln t - 2e^t + c \qquad [1]$$

$$\Rightarrow \ln x = \ln\left(\frac{x}{y}\right) - 2e^{\frac{x}{y}} + c$$

$$\Rightarrow \ln y = -2e^{\frac{x}{y}} + c$$

At $x = 0, y = 1$

$$\Rightarrow 0 = -2e^0 + c$$

$$\Rightarrow c = 2$$

$$\ln y = 2 - 2e^{\frac{x}{y}} \qquad [1]$$

17. Given : $\frac{dy}{dx} = -\frac{x + y\cos x}{1+\sin x}$

$$\Rightarrow \frac{dy}{dx} + \frac{\cos x}{1+\sin x}y = -\frac{x}{1+\sin x}$$

$$\frac{dy}{dx} + P(x)\,y = Q(x) \qquad [1]$$

Integrating factor $\text{IF} = e^{\int\left(\frac{\cos x}{1+\sin x}\right)dx}$

$$= e^{\log(1+\sin x)} = 1+\sin x$$

$$y.\text{IF} = \int \text{IF}\ Q(x)\ dx + C$$

$$y.(1+\sin x) = \int -\frac{x}{1+\sin x}(1+\sin x)\,dx + c \qquad [1]$$

$$\Rightarrow y.(1+\sin x) = -\frac{x^2}{2} + c$$

At x = 0 , $y = 1$

$$1.(1+0) = c$$

$$\Rightarrow c = 1 \qquad [1]$$

So $y(1+\sin x) = -\frac{x^2}{2} + 1$ [1]

18. We have,

$$\left(1+x^2\right)\frac{dy}{dx} + y = \tan^{-1}x$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{1+x^2} = \frac{\tan^{-1}x}{1+x^2} \qquad (1) \qquad [1]$$

Clearly, it is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Where

$$P = \frac{1}{1+x^2}$$

$$Q = \frac{\tan^{-1}x}{1+x^2}$$

$$\therefore \text{I.F.} = e^{\int P\,dx}$$

$$= e^{\int\frac{1}{1+x^2}dx}$$

$$= e^{\tan^{-1}x} \qquad [1]$$

Multiplying both sides of (1) by

$e^{\tan^{-1}x}$, we get

$$e^{\tan^{-1}x}\left(\frac{dy}{dx} + \frac{y}{1+x^2}\right) = e^{\tan^{-1}x}\frac{\tan^{-1}x}{1+x^2}$$

$$\Rightarrow e^{\tan^{-1}x}\frac{dy}{dx} + e^{\tan^{-1}x}\frac{y}{1+x^2}$$

$$= e^{\tan^{-1}x}\frac{\tan^{-1}x}{1+x^2}$$

Integrating both sides w.r.t x, we get

$$e^{\tan^{-1}x}y = \int \frac{\tan^{-1}x \times e^{\tan^{-1}x}}{1+x^2}dx + C$$

$$\Rightarrow e^{\tan^{-1}x}y = I + C \qquad (2) \qquad [1]$$

Here,

$$I = \int \frac{\tan^{-1}x \times e^{\tan^{-1}x}}{1+x^2}dx$$

Putting $\tan^{-1}x = t$, we get

$$\Rightarrow \frac{1}{1+x^2}dx = dt$$

$$\therefore I = \int t e^t \, dt$$

$$= t\int e^t \, dt - \int\left[\frac{d}{dt}(t)\int e^t dt\right]dt$$

$$= te^t - e^t$$

$$= (t-1)e^t$$

$$= \left(\tan^{-1}x - 1\right)e^{\tan^{-1}x}$$

Substituting the value of I in (2), we get

$$e^{\tan^{-1}x}y = \left(\tan^{-1}x - 1\right)e^{\tan^{-1}x} + C$$

$$\Rightarrow y = \tan^{-1}x - 1 + Ce^{-\tan^{-1}x}$$

Here, $y = \tan^{-1}x - 1 + Ce^{-\tan^{-1}x}$ is the required solution.

19. Given : $e^x \tan y dx + \left(2 - e^x\right)\sec^2 y dy = 0,$

$$\left(2 - e^x\right)\sec^2 y dy = -e^x \tan y dx$$

$$\frac{\sec^2 y}{\tan y}dy = \frac{-e^x}{\left(2-e^x\right)}dx$$

Integrating both sides.

$$\int \frac{\sec^2 y}{\tan y}dy = \int \frac{-e^x}{\left(2-e^x\right)}dx \qquad (1) \qquad [1]$$

For left side let $\tan y = a$

$$\sec^2 y = \frac{da}{dy}$$

$$\sec^2 y dy = da \qquad (2) \qquad [1]$$

For right side let $2 - e^x = b$

$$-e^x = \frac{db}{dx}$$

$$-e^x dx = db \qquad (3)$$

From (1), (2) and (3) we get

$$\int \frac{1}{a}da = \int \frac{1}{b}db$$

$$\log|a| = \log|b| + \log|c|$$

$$\log|a| = \log|bc| \qquad [1]$$

$$\Rightarrow a = bc$$

Substituting the values of a and b we get

$$\tan y = c\left(2 - e^x\right)$$

Given $y = \frac{\pi}{4}$ when $x = 0$

$$\tan\frac{\pi}{4} = c\left(2 - e^x\right)$$

$$1 = c(2-1)$$

$$c = 1$$

Substituting in the above equation we get,

$$\tan y = 1\left(2 - e^x\right)$$

$$\tan y = 2 - e^x \qquad [1]$$

20. Given : $\frac{dy}{dx} + 2y\tan x = \sin x,$

As the equation is of the form $\frac{dy}{dx} + Py = Q$

So, $P = 2\tan x,\ Q = \sin x$

Finding integrating factors we get

$$\text{I.F} = e^{\int P dx}$$

$$\text{I.F} = e^{\int 2\tan x dx} = e^{2\log|\sec x|} = e^{\log\left|\sec^2 x\right|} = \sec^2 x$$

[1]

Solution is $y(\text{I.F}) = \int Q(\text{I.F})\,dx$

$$y\left(\sec^2 x\right) = \int \sin x\left(\sec^2 x\right)dx$$

$$y\sec^2 x = \int \sec x \tan x\, dx$$

$$y\sec^2 x = \sec x + c$$

$$y = \frac{\sec x + c}{\sec^2 x}$$ [1]

$$y = \cos x + C\cos^2 x$$

Given $y = 0$ when $x = \frac{\pi}{3}$

Thus, $0 = \cos\frac{\pi}{3} + C\cos^2\frac{\pi}{3}$

$$0 = \frac{1}{2} + C\left(\frac{1}{4}\right)$$ [1]

$$-\frac{1}{2} = C\left(\frac{1}{4}\right)$$

$C = -2$

Substituting the value of C we get,

$$y = \cos x - 2\cos^2 x$$ [1]

21. $x\frac{dy}{dx} + y = x\cos x + \sin x$

Dividing both sides by x, we have

$$\frac{dy}{dx} + \frac{y}{x} = \frac{x\cos x + \sin x}{x}$$

Comparing with $\frac{dy}{dx} + Py = Q$ [1]

$$\therefore P = \frac{1}{x}$$

$$Q = \frac{x\cos x + \sin x}{x}$$

I.F. $= e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log|x|} = x$ [1]

Hence the solution is

$$y(\text{I.F.}) = \int Q(\text{I.F.})\, dx$$

$$y(x) = \int \frac{(x\cos x + \sin x)}{x} \cdot x dx$$

$$xy = \int x\cos x dx + \int \sin x dx$$

Integrating by parts taking x as first function, we get

$$xy = x\cdot(\sin x) - \int 1\cdot \sin x dx + \int \sin x dx$$

$xy = x\sin x + C$ (1) [1]

Using $x = \frac{\pi}{2}; y = 1$

$$\frac{\pi}{2}\cdot 1 = \frac{\pi}{2}\sin\frac{\pi}{2} + C$$

$\frac{\pi}{2} = \frac{\pi}{2} + C$ $\therefore C = 0$

Putting the value of C in (1)

$$xy = x\sin x$$

Or $y = \sin x$ which is the particular solution of given differential equation. [1]

22. $\frac{dy}{dx} = 1 + x + y + xy$

Or $\frac{dy}{dx} = (1 + x)(1 + y)$

Or $\frac{dy}{1 + y} = (1 + x)\, dx$

$$\int \frac{dy}{1 + y} = \int (1 + x)\, dx$$ [1]

$$\log|1 + y| = x + \frac{x^2}{2} + C$$

Given $y = 0$ when $x = 1$

i.e. $\log|1 + 0| = 1 + \frac{1}{2} + C$

$\therefore$ The particular solution is

$$\log|1 + y| = x + \frac{x^2}{2} - \frac{3}{2}$$ [1]

or the answer can expressed as

$$\log|1 + y| = \frac{x^2 + 2x - 3}{2}$$

Or $1 + y = e^{\frac{x^2 + 2x - 3}{2}}$ [1]

Or $y = e^{\frac{x^2 + 2x - 3}{2}} - 1$ [1]

23. $2ye^{x/y}dx+\left(y-2xe^{x/y}\right)dy=0$

$$2ye^{x/y}dx=-\left(y-2xe^{x/y}\right)dy \quad [1]$$

$$\frac{dx}{dy}=\frac{\left(2xe^{x/y}-y\right)}{2xe^{x/y}}=g\left(\frac{x}{y}\right) \quad [1]$$

R.H.S Of differential equation (I) is of the from $g\left(\frac{x}{y}\right)$ and so it homogeneous. [1]

$$\left[\begin{array}{l} Let\, x=vy\Rightarrow\frac{x}{y}=v \\ \frac{dx}{dy}=v.1+y.\frac{dv}{dy}\end{array}\right] \quad [1]$$

$$v+y.\frac{dv}{dy}=\frac{2ve^{v}-1}{2e^{v}}$$

$$y.\frac{dv}{dy}=\frac{2ve^{v}-1}{2e^{v}}-v$$

$$y\frac{dv}{dy}=\frac{2ve^{v}-1-v2e^{v}}{2e^{v}}=\frac{-1}{2e^{v}} \quad [1]$$

$$\int 2e^{v}dv=-\int\frac{dy}{y}$$

$$2e^{v}=-\log|y|+C$$

$$2e^{\frac{x}{y}}+\log|y|=C \quad [1]$$

24. $\left(\tan^{-1}y-x\right)dy=\left(1+y^2\right)dx$

$$\Rightarrow\frac{dx}{dy}=\frac{\tan^{-1}y}{1+y^2}-\frac{x}{1+y^2}$$

$$\Rightarrow\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2} \quad [1]$$

$$\text{IF}=e^{\int\frac{1}{1+y^2}dy}$$

$$\text{IF}=e^{\tan^{-1}y} \quad [1]$$

$$x.\text{IF}=\int Q\, IF\, dy+c$$

$$\Rightarrow x.e^{\tan^{-1}y}=\int\frac{\tan^{-1}y}{\left(1+y^2\right)}.e^{\tan^{-1}y}dy+c$$

Put $\tan^{-1}y=t$

$$\Rightarrow x.e^{\tan^{-1}y}=\int t.e^{t}.dt+c \quad [1]$$

$$\Rightarrow x.e^{\tan^{-1}y}=\left(t.e^{t}\right)-\left(e^{t}\right)+c$$

$$\Rightarrow x.e^{\tan^{-1}y}=\tan^{-1}y.e^{\tan^{-1}y}-e^{\tan^{-1}y}+c \quad [1]$$

Given: $x=0$, $y=0$

Substituting we get,

$$\Rightarrow (0).e^{\tan^{-1}(0)}=\tan^{-1}(0).e^{\tan^{-1}(0)}-e^{\tan^{-1}(0)}+c$$

[1]

$$\Rightarrow 0=0-e^{(0)}+c$$

$$\Rightarrow c=1$$

The equation is

$$\Rightarrow x.e^{\tan^{-1}y}=\tan^{-1}x.e^{\tan^{-1}y}-e^{\tan^{-1}y}+1 \quad [1]$$

25. $(x-y)\frac{dy}{dx}=(x+2y)$

$$\frac{dy}{dx}=\frac{x+2y}{x-y}$$

Let $y=Vx$ [1]

$$x\frac{dy}{dx}=V+x\frac{dV}{dx}$$

$$\Rightarrow V+x\frac{dV}{dx}=\frac{x+2(Vx)}{x-Vx} \quad [1]$$

$$\Rightarrow V+x\frac{dV}{dx}=\frac{1+2V}{1-V}$$

$$\Rightarrow x\frac{dV}{dx}=\frac{1+2V-V+V^2}{1-V}$$

$$\Rightarrow\int\frac{1-V}{1+V+V^2}dV=\int\frac{dx}{x}$$

$$\Rightarrow -\frac{1}{2}\int\frac{(2V+1)-3}{1+V+V^2}dV=\int\frac{dx}{x} \quad [1]$$

$$\Rightarrow -\frac{1}{2}\log\left|1+V+V^2\right|+\frac{3}{2}\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{V+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$=\log|x|+C$$

$$\Rightarrow -\frac{1}{2}\log\left|1+\frac{y}{x}+\frac{y^2}{x^2}\right|+\sqrt{3}\tan^{-1}\left(\frac{2\frac{y}{x}+1}{\sqrt{3}}\right)$$ [1]

$$=\log|x|+C$$

we have $y = 0$ when $x = 1$

$$\Rightarrow 0+\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=0+C$$

$$\Rightarrow C=\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$ [1]

$$\Rightarrow -\frac{1}{2}\log\left|1+\frac{V}{x}+\frac{V^2}{x^2}\right|+\sqrt{3}\tan^{-1}\left(\frac{2\frac{y}{x}+1}{\sqrt{3}}\right)$$

$$=\log|x|+\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$ [1]

26. Dividing Numerator and denominator by x^2, we have,

$$\frac{dy}{dx}=\frac{\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}$$

Let $y=vx\Rightarrow v=\frac{y}{x}$

$$\frac{dy}{dx}=v.1+x\frac{dy}{dx}$$ [1]

$$v+x\frac{dv}{dx}=\frac{v}{1+v^2}$$

$$x\frac{dv}{dx}=\frac{v}{1+v^2}-v$$

$$x\frac{dv}{dx}=\frac{v-v-v^3}{1+v^2}$$ [1]

$$x\frac{dv}{dx}=\frac{-v^3}{1+v^2}$$

$$\frac{1+v^2}{v^3}dv=-\frac{dx}{x}$$

$$\int\left(\frac{1}{v^3}+\frac{v^2}{v^3}\right)dv=-\int\frac{dx}{x}$$ [1]

$$\frac{v^{-2}}{-2}+\log|v|=-\log|x|+c$$

$$\frac{-1}{2\left(\frac{y}{x}\right)^2}+\log\left|\frac{y}{x}\right|+\log|x|=c$$

$$\frac{-x^2}{2y^2}+\log\left|\frac{y}{x}\times x\right|=c$$ [1]

$$\frac{-x^2}{2y^2}+\log|y|=c$$

$\because\ y=1, x=0, \log 1=0$

$$=\frac{0}{2(1)^2}+\log 1=C$$

$\therefore\ C=0$

Putting the value of C in equation (i)

$$\frac{-x^2}{2y^2}+\log|y|=0$$ [1]

$$\frac{-x^2+2y^2\log|y|}{2y^2}=0$$

$$2y^2\log|y|-x^2=0$$

which is the particular solution of the given differential equation. [1]

Smart Notes ....

Smart Notes ....

CHAPTER 10

Vector Algebra

Chapter Analysis of Previous Three Years' Board Exams

Number of Questions Asked in Exams

Topics	2018		2017		2016	
	Delhi	All India	Delhi	All India	Delhi	All India
Coplanarity		4 marks		4 marks	4 marks	
Mutually Perpendicular Unit Vectors	4 marks	4 marks	4 marks			
Diagonal Vectors	1 mark	1 mark				
Length of Unit Vectors						1 mark
Position Vector					1 mark	1 mark
Vector Equation of Line			2 marks			
Length of Median					1 mark	
Diagonal Vectors						4 marks
Area of Triangle				4 marks		
Angle between Vectors	2 marks	2 marks				

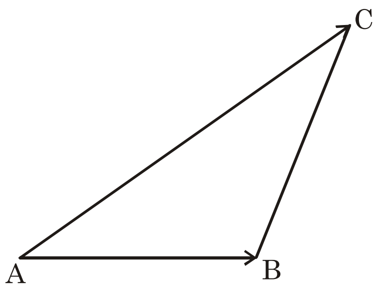
[TOPIC 1] Algebra of Vectors

Summary

- A vector quantity has both magnitude and direction where the magnitude is a distance between the initial and terminal point of the vector. Let's assume a vector starts at a point A and ends at a point B. Therefore the magnitude of the vector is denoted by $|\overrightarrow{AB}|$.
- $\overrightarrow{OA} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector of any point $A(x, y, z)$ having a magnitude equal to $\sqrt{x^2 + y^2 + z^2}$. Where O is the origin (0, 0, 0) and P is any point in the space.
- The angles α, β, γ are known as the direction angles which are made by the position vector and the positive x, y, z – axes respectively and their cosine values ($\cos\alpha$, $\cos\beta$, $\cos\gamma$) are known as direction cosines, denoted by l, m, n respectively.
- The projections of a vector along the respective axes are represented by the direction ratios which are the scalar components of the vector. They are denoted by a, b, c respectively.
- The direction cosines, direction ratios, and magnitude of a vector are related as:

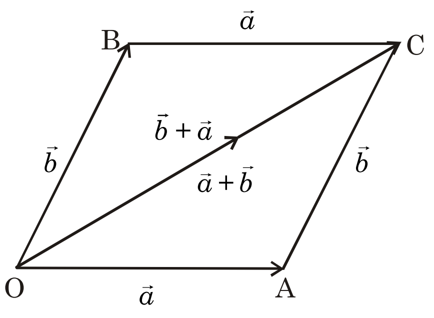
$$l = \frac{a}{r}, m = \frac{b}{r}, n = \frac{c}{r}$$

- In general, $l^2 + m^2 + n^2 = 1$ but $a^2 + b^2 + c^2 \neq 1$.
- Zero vector (also known as a null vector) is symbolized by $\vec{0}$. Its initial and terminal points coincide.
- A unit vector has a magnitude equal to 1 and is denoted by $\hat{a}$.
- If two or more than two vectors have the same initial points, they are called as co-initial vectors.
- The vectors which are parallel to the same line are known as collinear vectors.
- The vectors having equal magnitude and same direction are called equal vectors.
- A vector having the same magnitude as the given vector but opposite direction is known as the negative of the given vector.
- **Triangle law of vector addition:** Let's say that A, B, and C are the vertices of a triangle then



$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

- **Parallelogram law of vector addition:** If two vectors are represented by the two adjacent sides of a parallelogram, then their sum is represented by the diagonal of that parallelogram through their common point. For example, the



$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$

$\Rightarrow \overrightarrow{OC} = \vec{a} + \vec{b}$

- Vector addition is commutative as well as associative in nature and also has zero vector as an additive identity.
- The multiplication of any vector $\vec{a}$ by a scalar λ is denoted by $\lambda\vec{a}$ and has the same direction as the original vector if λ is positive and opposite direction if λ is negative. Its magnitude is $|\lambda\vec{a}| = |\lambda||\vec{a}|$.

- The unit vector of any vector $\vec{a}$ in its direction is written as $\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$
- The unit vectors along the positive x, y, z axes are denoted by $\hat{i}, \hat{j}, \hat{k}$ respectively.
- **Component form of a vector:** The component form of any vector is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ where x, y, z are called as the scalar components and $x\hat{i}, y\hat{j}, z\hat{k}$ as the vector components of $\vec{r}$. x, y, z is also called the rectangular components.
- If two vectors are in their component form as $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then
  - The sum of the vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$.
  - The difference of the vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$
  - The vectors $\vec{a}$ and $\vec{b}$ are equal if $a_1 = b_1, a_2 = b_2$ and $a_3 = b_3$.
  - The multiplication of a vector $\vec{a}$ by scalar λ is given by $\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$.
- **Vector joining two points:** The magnitude of a vector $\overrightarrow{A_1A_2}$ joining two points $\overrightarrow{A_1}(x_1, y_1, z_1)$ and $\overrightarrow{A_2}(x_2, y_2, z_2)$ is

$$\overrightarrow{A_1A_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- **Section Formula:** The position vector of a point C dividing the line segment joining two points A and B (having position vectors $\vec{a}, \vec{b}$ respectively) in the ratio of $m : n$
  - Internally: $\vec{r} = \frac{m\vec{b} + n\vec{a}}{m + n}$
  - Externally: $\vec{r} = \frac{m\vec{b} - n\vec{a}}{m - n}$
  - If C is the midpoint of A and B, then $\vec{r} = \frac{\vec{a} + \vec{b}}{2}$

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

1 Mark Questions

1. If $\vec{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ and $\vec{b} = 3\hat{i} - y\hat{j} + \hat{k}$ are two equal vectors, then write the value of x + y + z

[DELHI 2013]

2. If a unit vector $\vec{a}$ makes angles $\frac{\pi}{3}$ with $i, \frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ *with* $\hat{k}, [0,1]$ then find the value of θ.

[DELHI 2013]

3. Find the Cartesian equation of the line which passes through the point (–2, 4, –5)and is the parallel to the line $\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}$.

[DELHI 2013]

4. If a line makes angles 90°.60°and θ with x, y and z-axis repectively, where θ is acute, then find θ.

[DELHI 2015]

5. If a line makes angle 90° and 60° respectively with the positive direction of x and y axes, find the angle which it makes with the positive direction of z – axis.

[DELHI 2017]

6. Find the vector equation of the line passing through the point $A(1,2,-1)$ and parallel to the line $5x - 25 = 14 - 7y = 35z$.

[DELHI 2017]

7. Find the position vector of a point which divides the join of points with position vectors $\vec{a}-2\vec{b}$ and $2\vec{a}+\vec{b}$ externally in the ratio $2:1$

[DELHI 2016]

8. The two vectors $\hat{j}+\hat{k}$ and $3\hat{i}-\hat{j}+4\hat{k}$ represent the two sides AB and AC, respectively of a ΔABC. Find the length of the median through point A.

[DELHI 2016]

9. If a line has direction ratios 2, –1, –2 then what are its direction?

[DELHI 2012]

10. Find the sum of the vectors

$\vec{a}=\hat{i}-2\hat{j}+\hat{k}$

$\vec{b}=-2\hat{i}+4\hat{j}+5\hat{k}$ and

$\vec{c}=\hat{i}-6\hat{j}-7\hat{k}$

[ALL INDIA 2012]

11. For what value of $\vec{a}$ the vectors

$2\hat{i}-3\hat{i}+4\hat{k} and\, a\hat{i}+6\hat{i}-8\hat{k}$ are collinear?

[DELHI 2011]

12. Write the position vector of the mid-point of vector joining the points P(2,3,4) and Q(4,1,–2).

[ALL INDIA 2011]

13. P and Q are two points with position vectors $3\vec{a}-2\vec{b}$ and $\vec{a}+\vec{b}$ respectively. Write the position vector of point R which divides the line segment PQ in the ratio 2 : 1 externally.

[ALL INDIA 2013]

14. If $\vec{a}=\hat{i}+2\hat{j}-k, \vec{b}=2\hat{i}+\hat{j}+k$ and $\vec{c}=5\hat{i}-4\hat{j}+3k$, then find the value of $\left(\vec{a}+\vec{b}\right).\vec{c}$

[DELHI 2015]

15. Write the direction ratios of the following line:

$$x=-3, \frac{y-4}{3}=\frac{2-z}{1}$$

[ALL INDIA 2015]

16. If $\vec{a}=4\hat{i}-\hat{j}+\hat{k}$ and $\vec{b}=2\hat{i}-2\hat{j}+\hat{k}$, then find a unit vector parallel to the vector $\vec{a}+\vec{b}$.

[ALL INDIA 2016]

Solutions

1. Given : $\vec{a}=\vec{b}$

$x\hat{i}+2\hat{j}-z\hat{k}=3\hat{i}-y\hat{j}+\hat{k}$

$x=3 \qquad -y=2, \qquad -z=1$

$\Rightarrow x=3, \qquad y=-2, \qquad z=-1$

$x+y+z=3+(-2)+(-1)=0$ [1]

2. Given : Here, $l=\cos a=\cos\frac{\pi}{3}=\frac{1}{2}$

$m=\cos\beta=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}, n=\cos\theta$

As we know , $l^2+m^2+n^2=1$

$\left(\frac{1}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2+\cos^2\theta=1$ [½]

$\cos^2\theta=1-\frac{1}{4}-\frac{1}{2}=\frac{4-1-2}{4}=\frac{1}{4}$

$\cos^2=\frac{1}{2} \qquad \Rightarrow \theta=\frac{\pi}{3}$ [½]

3. Given line is $\frac{x+3}{3}=\frac{4-y}{5}=\frac{z+8}{6}$

$\Rightarrow \frac{x+3}{3}=\frac{y-4}{-5}=\frac{z+8}{6}$

Direction ratio of the given line <3, –5, 6>

Direction ratios of the required lines are <3, –5, 6>

Passing through point (–2, 4 – 5)

Cartesian equations of the line are

$\frac{x+2}{3}=\frac{y-4}{-5}=\frac{z+5}{6}$ [1]

4. Here, $\alpha=90^o, \beta=60^o, Let \delta=\theta$

$l=\cos\alpha=\cos 90^o=0$

$m=\cos\beta=\cos 60^o=\frac{1}{2}$ [½]

$n=\cos\delta=\cos\theta$

$l^2+m^2+n^2=1$

$(0)^2+\left(\frac{1}{2}\right)^2+\cos^2\theta=1$

$\cos^2\theta = 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore \cos\theta = \frac{\sqrt{3}}{2} = \cos 30^o$

$\theta = 30^o \, or \frac{\pi}{6}$ $(\because \theta$ is an acute angle$)$ [½]

5. Let the required angle makes with the z - axis be θ

Using formula $l^2 + m^2 + n^2 = 1$

$\cos^2 90° + \sin^2 60° + \cos^2\theta = 1$

$0 + \left(\frac{1}{2}\right)^2 + \cos^2\theta = 1$

$\cos^2\theta = 1 - \frac{1}{4}$

$\cos^2\theta = \frac{3}{4}$

$\therefore \quad \cos\theta = \frac{\pm\sqrt{3}}{2}$

$\therefore \ \theta = 30° \ or \ 150°$ [1]

6. Given parallel line

$5x - 25 = 14 - 7y = 35z$

$5(x-5) = -7(y-2) = 35z$

Dividing all sides by 35 we get

$\frac{5(x-5)}{35} = \frac{-7(y-2)}{35} = \frac{35z}{35}$

$\frac{(x-5)}{7} = \frac{(y-2)}{-5} = \frac{z}{1}$ ...(1) [1]

Direction ratios of parallel line are 7, –5, 1.

Thus the direction ratio of line are 7, –5, 1.

Vector equation of line : $\vec{r} = \vec{a} + \lambda\vec{b}$

The direction ratio of line are 7, –5, 1.

$\Rightarrow \vec{b} = 7\hat{i} - 5\hat{j} + \hat{k}$

Passing point $(1, 2, -1)$ $\Rightarrow \vec{b} = 7\hat{i} - 5\hat{j} + \hat{k}$

$\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(7\hat{i} - 5\hat{j} + \hat{k})$

7. The required position vector is:

$\frac{2(2\vec{a} + \vec{b}) - 1(\vec{a} - 2\vec{b})}{2 - 1}$

$\frac{4\vec{a} + 2\vec{b} - \vec{a} + 2\vec{b}}{1} = 3\vec{a} + 4\vec{b}$ [1]

8. In ΔABC, by Δ law of vector addition

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$(\hat{j} + \hat{k}) + \overrightarrow{BC} = 3\hat{i} - \hat{j} + 4\hat{k}$

$\overrightarrow{BC} = 3\hat{i} - \hat{j} + 4\hat{k} - \hat{j} - \hat{k}$

$= 3\hat{i} - 2\hat{j} + 3\hat{k}$

D is the mid-point of $\overrightarrow{BC}$

$(\because AD$ is median$)$

$\therefore \quad \overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC} = \frac{3}{2}\hat{i} - \frac{2}{2}\hat{j} + \frac{3}{2}\hat{k}$

$= \frac{3}{2}\hat{i} - \hat{j} + \frac{3}{2}\hat{k}$ [½]

In ΔABD,

$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$

(By Δ law of vector addition)

$= \hat{j} + \hat{k} + \frac{3}{2}\hat{i} - \hat{j} + \frac{3}{2}\hat{k}$

$= \frac{3}{2}\hat{i} + \frac{5}{2}\hat{k}$

$|\overrightarrow{AD}| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2}$

$= \sqrt{\frac{9}{4} + \frac{25}{4}}$

$= \sqrt{\frac{34}{4}}$

Length of median, $AD = \frac{\sqrt{34}}{2}$ units [½]

9. Given $a = 2, b = -1, c = -2$

$\sqrt{a^2 + b^2 + c^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$

Direction cosines are

$l = \frac{2}{3}, m = \frac{-1}{3}, \text{n} = \frac{-2}{3}$ [1]

10. Given : $\vec{a} + \vec{b} + \vec{c}$

$= (\hat{i} - 2\hat{j} + \hat{k}) + (-2\hat{i} + 4\hat{j} + 5\hat{k}) + (\hat{i} - 6\hat{j} - 7\hat{k})$

$= 0\hat{i} - 4\hat{j} - \hat{k}$ [1]

11. Given $2\hat{i}-3\hat{j}+4\hat{k}$ and $a\hat{i}+6\hat{j}-8\hat{k}$ are collinear

$\left(2\hat{i}-3\hat{j}+4\hat{k}\right)=\lambda\left(a\hat{i}+6\hat{j}-8\hat{k}\right)$, where λ is a scalar $2\hat{i}-3\hat{i}+4\hat{k}=\lambda a\hat{i}+6\lambda\hat{j}-8\lambda\hat{k}$

$\therefore\ -3=6\lambda$

$\therefore\ \lambda=-\dfrac{1}{2}$

Now $2=\lambda a$

$\therefore 2=\left(\dfrac{-1}{2}\right)a$

$\therefore -a=+4$ [1]

$\therefore a=-4$

12. $P=(2,3,4)$

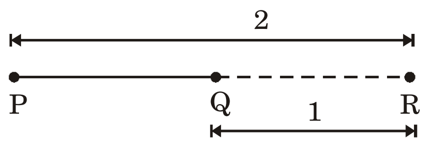
$Q=(4,1,-2)$

$$\overline{OR}=\frac{\left(2\hat{i}+3\hat{j}+4\hat{k}\right)+(4\hat{i}+\hat{j}-2\hat{k})}{2}$$

$$=\frac{6\hat{i}+4\hat{j}+2\hat{k}}{2}$$

$=3\hat{i}+2\hat{j}+\hat{k}$ [1]

13. P.V. of P is $3\vec{a}-2\vec{b}$



P.V. of Q is $\vec{a}+\vec{b}$

Point R divides segment PQ in ratio 2:1 externally.

$$\text{P.V. of R}=\frac{(P.V. of\ p)1-(P.V. of\ Q)(2)}{1-2}$$ [½]

$$\text{P.V. of R}=\frac{\left(3\vec{a}-2\vec{b}\right)(1)-\left(\vec{a}+\vec{b}\right)(2)}{1-2}=\frac{\vec{a}-4\vec{b}}{-1}$$

P.V. of R = 4 $\vec{b}-\vec{a}$ [½]

14. *Given* : $\vec{a}=\hat{i}+2\hat{j}-\hat{k}, \vec{b}=2\hat{i}+\hat{j}+\hat{k}$ and

$\vec{c}=5\hat{i}-4\hat{j}+3\hat{k}$

$(\vec{a}+\vec{b})=\hat{i}+2\hat{j}-\hat{k}+2\hat{i}+\hat{j}+\hat{k}=3\hat{i}+3\hat{j}$ [½]

$\left(\vec{a}+\vec{b}\right).\vec{c}=\left(3\hat{i}+3\hat{j}\right)\left(5\hat{i}-4\hat{j}+3\hat{k}\right)$

$\left(\vec{a}+\vec{b}\right).\vec{c}=3$ [½]

15. Given

$x=-3, \dfrac{y-4}{3}=\dfrac{2-z}{1}\qquad \dfrac{x+3}{0}=\dfrac{y-4}{3}=\dfrac{z-2}{-1}$ [1]

Direction ratios are 0, 3, –1

16. $\vec{a}+\vec{b}=4\hat{i}-\hat{j}+\hat{k}+2\hat{i}-2\hat{j}+\hat{k}$

$=6\hat{i}-3\hat{j}+2\hat{k}$

Unit vector parallel to

$$\hat{a}+\hat{b}=\frac{6\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{36+9+4}}$$

$$=\frac{6\hat{i}-3\hat{j}+2\hat{k}}{7}$$ [1]

[TOPIC 2] Product of Two Vectors, Scalar Triple Product

Summary

- **Scalar (or dot) product of two vectors $\vec{a}$ and $\vec{b}$:** The scalar product of two vectors having an angle θ between them is denoted by $\vec{a}\cdot\vec{b}$ and is defined by

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

or $\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$

- If $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a}\perp\vec{b}$
- Properties of scalar product:
  - Let $\vec{a},\vec{b}$ and $\vec{c}$ be any three vectors then $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$
  - Let $\vec{a}$ and $\vec{b}$ be any two vectors, and λ be any scalar. Then $(\lambda\vec{a})\cdot\vec{b} = \lambda(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(\lambda\vec{b})$
- Projection of a vector $\vec{a}$ on another vector $\vec{b}$ is given as $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)$.
- Projection of a vector $\vec{b}$ on another vector $\vec{a}$ is given as $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)$.
- **Vector (or cross) product of two vectors $\vec{a}$ and $\vec{b}$:** The vector product of two vectors having an angle θ between them is denoted by $\vec{a}\times\vec{b}$ and is defined by:

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\hat{n}$$

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$$

- If $\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow \vec{a}\parallel\vec{b}$
- If the vectors are in their component form as $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then their cross product is given by

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

  The dot product is given by

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

- Properties of vector product:
  - Let $\vec{a},\vec{b}$ and $\vec{c}$ be any three vectors then $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$.
  - Let $\vec{a}$ and $\vec{b}$ be any two vectors, and λ be any scalar. Then $(\lambda\vec{a})\times\vec{b} = \lambda(\vec{a}\times\vec{b}) = \vec{a}\times(\lambda\vec{b})$
- Scalar triple product of three vectors $[A, B, C]$ is given by $\vec{A}.(\vec{B}\times\vec{C})$ which is the volume of a parallelepiped whose sides are given by vectors $\vec{A},\vec{B}$ and $\vec{C}$.

PREVIOUS YEARS' EXAMINATION QUESTIONS

TOPIC 2

1 Mark Questions

1. $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}+\vec{b}|=|\vec{a}|$, then prove that vector $2\vec{a}+\vec{b}$ is perpendicular to vector $\vec{b}$.

[DELHI 2013]

2. If $\vec{a}=7\hat{i}+\hat{j}-4\hat{k}$ and $\vec{b}=2\hat{i}+6\hat{j}+3\hat{k}$, then find the projection of $\vec{a}$ and $\vec{b}$

[DELHI 2015]

3. Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $2\hat{i}-3\hat{j}+6\hat{k}$.

[DELHI 2014]

4. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+\vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$.

[DELHI 2014]

5. Find λ' when the projection of $\vec{a}=\lambda\hat{i}+\hat{j}+4\hat{k}$ on $\vec{b}=2\hat{i}+6\hat{j}+3\hat{k}$ is 4 units.

[DELHI 2012]

6. Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{9}{2}$.

[DELHI 2018]

7. If $\vec{a}.\vec{a}=0$ and $\vec{a}.\vec{b}=0$ then what can be concluded about the vector $\vec{b}$?

[ALL INDIA 2011]

8. Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=15$.

[ALL INDIA 2013]

9. Find the value of 'p' for which the vectors $3\hat{i}+2\hat{j}+9\hat{k}$ and $\hat{i}-2p\hat{j}+3\hat{k}$ are parallel.

[ALL INDIA 2014]

10. Find $\vec{a}.(\hat{b}\times\hat{c})$, if $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}+2\hat{k}$.

[ALL INDIA 2014]

11. Write the value of $\vec{a}(\vec{b}\times\vec{a})$.

[ALL INDIA 2015]

12. Find λ and μ if

$$(\hat{i}+3\hat{j}+9\hat{k})X(3\hat{i}-\lambda\hat{j}+\mu\hat{k})=\vec{0}$$

[ALL INDIA 2016]

4 Marks Questions

13. If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, find $(\vec{r}\times\hat{i})\cdot(\vec{r}\times\hat{j})+xy$

[DELHI 2015]

14. If $\vec{a},\vec{b},\vec{c}$ are mutually perpendicilar vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a},\vec{b}$ and $\vec{c}$. Also, find the angle which $\vec{a}+\vec{b}+\vec{c}$ make with $\vec{a}$ or $\vec{b}$ or $\vec{c}$

[DELHI 2017]

15. Find the vector and Cartesian equations of the line through the point $(1,2,-4)$ and perpendicular to the two lines.

$$\vec{r}=(8\hat{i}-19\hat{j}+10\hat{k})+\lambda(3\hat{i}-16\hat{j}+7\hat{k}) \text{ and}$$
$$\vec{r}=(15\hat{i}+29\hat{j}+5\hat{k})+\mu(3\hat{i}+8\hat{j}-5\hat{k})$$

[DELHI 2016]

16. Prove that, for any three vectors $\vec{a},\vec{b},\vec{c}$

$$[\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}]=2[\vec{a},\vec{b},\vec{c}]$$

[DELHI 2014]

17. Vectors $\vec{a},\vec{b},\vec{c}$ are such that $\vec{a}+\vec{b}+\vec{c}=0$ and $|\vec{a}|=3,|\vec{b}|=5$ *and* $|\vec{c}|=7$. Find the angle between $\vec{a}$ and $\vec{b}$.

[DELHI 2014]

18. If $\vec{a},\vec{b},\vec{c}$ are three vectors such that $|\vec{a}|=5,|\vec{b}|=12$ and $|\vec{c}|=13$ and $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ find the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$.

[ALL INDIA 2012]

19. Find a unit vector perpendicular to of each of the vectors $\vec{a}+\vec{b}\ \ \vec{a}-\vec{b},\ \ \vec{a}+\vec{b}\,and\,\vec{a}-\vec{b}$, where $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$.

[DELHI 2011]

20. Find the angle between the following pair of lines:

$$\frac{-x+2}{-2}=\frac{y-1}{7}=\frac{z+3}{-3}\text{ and }\frac{x+2}{-1}=\frac{2y-8}{4}=\frac{z-5}{4}$$

And check whether the line are parallel or perpendicular

[DELHI 2011]

21. Let $\vec{a}=4\hat{i}+5\hat{j}-\hat{k},\ \vec{b}=\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}-\hat{k}$

Find a vector $\vec{d}$ which is perpendicular to both $\vec{c}$ and $\vec{b}$ and $\vec{d}\cdot\vec{a}=21$

[DELHI 2018]

22. Show that each of the given three vectors is a unit vector:

$$\frac{1}{7}\left(2\hat{i}+3\hat{j}+6\hat{k}\right),\frac{1}{7}\left(3\hat{i}-6\hat{j}+2\hat{k}\right)\text{ and }\frac{1}{7}\left(6\hat{i}+2\hat{j}-3\hat{k}\right)$$

also show that they are mutually perpendicular to each other.

[ALL INDIA 2011]

23. If $\vec{a}=\hat{i}-\hat{j}+7\hat{k}$ and $\vec{b}=5\hat{i}-\hat{j}+\lambda\hat{k}$, then find the value of λ, so that $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular vectors.

[ALL INDIA 2013]

24. Show that the lines

$$\vec{r}=3\hat{i}+2\hat{j}-4\hat{k}+\lambda\left(\hat{i}+2\hat{j}+2\hat{k}\right)$$

$\vec{r}=5\hat{i}-2\hat{j}+\mu\left(3\hat{i}+2\hat{j}+6\hat{k}\right)$ are intersecting. Hence find their point of intersection.

[ALL INDIA 2013]

25. Find the vector equation of the plane through the points (2, 1, –1) and (–1, 3, 4) and perpendicular to the plane $x-2y+4z=10$

[ALL INDIA 2013]

26. A line passes through (2, –1, 3) and is perpendicular to the lines

$$\vec{r}=\left(2\hat{i}-\hat{j}-3\hat{k}\right)+\quad\left(\hat{i}+2\hat{j}+2\hat{k}\right)$$

Obtain its equation in vector and Cartesian form.

[ALL INDIA 2014]

27. Show that four points A, B, C and D whose position vectors are $4\hat{i}+5\hat{j}+\hat{k},-\hat{j}-\hat{k}\ \ 3\hat{i}+9\hat{j}+4\hat{k}$ and $4\left(-\hat{i}+\hat{j}+\hat{k}\right)$ respectively are coplanar.

[ALL INDIA 2015]

28. Given vectors: $2\hat{i}-\hat{j}+\hat{k},\hat{i}-3\hat{j}-5\hat{k},3\hat{i}-4\hat{j}-4\hat{k}$ are the position vectors of A,B and C respectively. Find the area of triangle ABC.

[ALL INDIA 2017]

29. Find the value of λ, if four points with position vectors $3\hat{i}+6\hat{j}+9\hat{k},\hat{i}+2\hat{j}+3\hat{k},2\hat{i}+3\hat{j}+\hat{k}$ and $4\hat{i}+6\hat{j}+\lambda\hat{k}$ are coplanar.

[ALL INDIA 2017]

6 Marks Question

30. If θ is the angle between two vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $\widehat{3i}-2\hat{j}+\hat{k}$

[DELHI 2018]

Solutions

1.

$$\left|\vec{a}+\vec{b}\right|=\left|\vec{a}\right|$$
$$\left|\vec{a}+\vec{b}\right|^2=\left|\vec{a}\right|^2$$
$$\left(\vec{a}+\vec{b}\right)^2=\vec{a}^2$$
$$\left(\vec{a}+\vec{b}\right).\left(\vec{a}+\vec{b}\right)=\vec{a}^2$$
$$\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b}=\vec{a}^2$$
$$\vec{a}^2+\vec{a}.\vec{b}+\vec{a}.\vec{b}+\vec{b}^2-\vec{a}^2=0$$

$2\vec{a}\vec{b}+\vec{b}^2=0$ [1]

2. $\vec{a}.\vec{b}=\left(7\hat{i}+\hat{j}-4\hat{k}\right).\left(2\hat{i}+6\hat{j}+3\hat{k}\right)$

$=14+6-12=8$

$\left|\vec{b}\right|=\sqrt{4+36+9}=\sqrt{49}=7$

The projection of $\vec{a}$ on $\vec{b}=\dfrac{\vec{a}.\vec{b}}{\left|\vec{b}\right|}=\dfrac{8}{7}$ [1]

3. Let the given vectors be $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and

$\vec{b} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

Projection of $\vec{a}$ on

$\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{2-9+42}{\sqrt{4+9+36}} = \frac{35}{7} = 5$ [1]

4. $|\vec{a}| = |\vec{b}| = |\vec{a}+\vec{b}| = 1$

$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})^2$

$(1)^2 = (\vec{a}+\vec{b})(\vec{a}+\vec{b})$

$1 = \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b},$

$1 = |\vec{a}|^2 + \vec{a}.\vec{b} + \vec{b}.\vec{a} + |\vec{b}|^2$

$1 = (1)^2 + 2\vec{a}.\vec{b} + (1)^2$ [½]

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

$1 - 1 - 1 = 2|\vec{a}||\vec{b}|\cos\theta, -1 = 2(1)(1)\cos\theta$

$-\frac{1}{2} = \cos\theta \Rightarrow -\cos\frac{\pi}{3} = \cos\theta$

$\therefore \theta = \frac{2\pi}{3}$ [½]

5. Projection of $\vec{a}$ on $\vec{b} = 4$ units

$\frac{\vec{a}.\vec{b}}{|\vec{b}|} = 4$

$\frac{(\lambda\hat{i} + \hat{j} + 4\hat{k}).(2\hat{i} + 6\hat{j} + 3\hat{k})}{\sqrt{4+36+9}} = 4$

$\frac{2\lambda + 6 + 12}{7} = 4$

$2\lambda + 18 = 28$

$2\lambda = 10$

$\lambda = 5$ [1]

6. Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$.

It is given that $|\vec{a}| = |\vec{b}|, \vec{a}.\vec{b} = \frac{9}{2}$ and $\theta = 60°$ (1)

We know that $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ [½]

$\therefore \frac{9}{2} = |\vec{a}||\vec{a}| \cos 60°$ (Using 1)

$\Rightarrow \frac{9}{2} = |\vec{a}|^2 \times \frac{1}{2}$

$\Rightarrow |\vec{a}|^2 = 9$

$|\vec{a}| = |\vec{b}| = 3$ [½]

7. $\vec{a}.\vec{a} = 0$

$\vec{a}.\vec{b} = 0$

Now,

$\vec{a}.\vec{a} = 0$

$|\vec{a}|^2 = 0$

$\vec{a} = 0$

Thus, $\vec{a}$ is a zero vector and $\vec{b}$ can be any vector. [1]

8. Given $|\vec{a}| = 1$

$(\vec{x} - \vec{a}).(\vec{x} + \vec{a}) = 15$

$|\vec{x}|^2 - |\vec{a}|^2 = 15$

$|\vec{x}|^2 - 1 = 15$

$|\vec{x}|^2 = 16$

$|\vec{x}| = 4$ [1]

9. Let $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}, \vec{b} = \hat{i} - 2p\hat{j} + 3\hat{k}$

If $\vec{a}, \vec{b}$ are parallel vector then their exist a, λ such that $\vec{a} = \lambda\vec{b}$

So $3\hat{i} + 2\hat{j} + 9\hat{k} = \lambda(\hat{i} - 2p\hat{j} + 3\hat{k})$

Compare $3 = \lambda$ $2 = -2p\lambda$ $9 = 3\lambda \Rightarrow \lambda = 3$

Put value of λ *in Eq.*$(2) = -2p\lambda$

$2 = -2p.3$

$p = -1/3$ [1]

10. If $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}, \vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and

$\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$

Then $\vec{a}.(\vec{b} \times \vec{c}) \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$ [½]

Expand along

$R_1 = 2[4-1] - 1[-2-3] + 3[-1-6]$

$= 6 + 5 - 21 = -10$ [½]

11. $\left[\vec{a}\left(\vec{b}\times\vec{a}\right)\right]=\left[\vec{a}\quad\vec{b}\quad\vec{a}\right]=0$ [1]

12. Given, $\left(\hat{i}+3\hat{j}+9\hat{k}\right)X\left(3\hat{i}-\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}$

$\Rightarrow\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 3 & 9\\ 3 & -\lambda & \mu\end{vmatrix}=\vec{0}$ [½]

$\Rightarrow\hat{i}(3\mu+9\lambda)-\hat{j}(\mu-27)+\hat{k}(-\lambda-9)$

$=0\hat{i}+0\hat{j}+0\hat{k}$

$-\lambda-9=0$

$\lambda=-9$

Put $\lambda=-9$

$3\mu+9\lambda=0$

$\Rightarrow\mu=27.$ [½]

13. $\vec{r}\times\hat{i}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x & y & z\\ 1 & 0 & 0\end{vmatrix}$

$=\hat{1}(0)-\hat{j}(-z)+\hat{k}(-y)=z\hat{j}-y\hat{k}$ …(i) [1]

$\vec{r}\times\hat{j}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x & y & z\\ 0 & 1 & 0\end{vmatrix}$ [2]

$=\hat{i}(-z)-\hat{j}(0)+\hat{k}(x)=-z\hat{i}+x\hat{k}$ … (ii)

$\left(\vec{r}\times\hat{i}\right)\cdot\left(\vec{r}\times\hat{j}\right)+xy=\left(z\hat{j}-y\hat{k}\right).\left(-z\hat{i}-x\hat{k}\right)+xy=0$

$+0-xy+xy=0$ [1]

(from (i) and (ii))

14. Let $|\vec{a}|=|\vec{b}|=|\vec{c}|=m$ (Let) ...(i)

Since $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors.

$\therefore\quad\vec{a}.\vec{b}=0,\quad\vec{b}.\vec{c}=0,\quad\vec{c}.\vec{a}=0$

$\left|\vec{a}+\vec{b}+\vec{c}\right|^2=\left(\vec{a}+\vec{b}+\vec{c}\right).\left(\vec{a}+\vec{b}+\vec{c}\right)\quad\left[\because|\vec{x}|^2=\vec{x}\vec{x}\right]$

$\Rightarrow\left|\vec{a}+\vec{b}+\vec{c}\right|^2$ [1]

$\Rightarrow\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{a}.\vec{c}+\vec{b}.\vec{a}+\vec{b}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}+\vec{c}.\vec{b}+\vec{c}.\vec{c}$

$\Rightarrow\left|\vec{a}+\vec{b}+\vec{c}\right|^2=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2$ ..... $\left[\begin{array}{l}\because\vec{a}.\vec{a}=|\vec{a}|^2\\ \vec{a}.\vec{b}=\vec{b}.\vec{a}=0\end{array}\right.$

$\Rightarrow\left|\vec{a}+\vec{b}+\vec{c}\right|^2=m^2+m^2+m^2$ [From(i)

$\Rightarrow\left|\vec{a}+\vec{b}+\vec{c}\right|^2=3m^2$

$\Rightarrow\left|\vec{a}+\vec{b}+\vec{c}\right|=m\sqrt{3}$ ...(ii) [1]

Let make angles respectively, then

$\cos\alpha=\dfrac{\vec{a}\left(\vec{a}+\vec{b}+\vec{c}\right)}{|\vec{a}|\left|\vec{a}+\vec{b}+\vec{c}\right|}\quad\left[\because\cos\theta=\dfrac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|}\right.$

$\cos\alpha=\dfrac{\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{a}.\vec{c}}{|\vec{a}|\left(m\sqrt{3}\right)}$ [From (*ii*)

$=\dfrac{|\vec{a}|^2+0+0}{|\vec{a}|\left(m\sqrt{3}\right)}=\dfrac{m^2}{m\left(m\sqrt{3}\right)}\quad\left[\begin{array}{l}\because\vec{a}.\vec{a}=|\vec{a}|^2\\ \vec{a}.\vec{b}=\vec{b}.\vec{a}=0\end{array}\right.$

$\cos\alpha=\dfrac{m^2}{m\left(m\sqrt{3}\right)}=\dfrac{1}{\sqrt{3}}$ [From(*i*)

$\alpha=\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ [1]

Similarly $\beta=\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

$\gamma=\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

$\therefore\alpha=\beta=\gamma=\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

Hence, $\vec{a},\vec{b},\vec{c}$ is equally inclined to and required angle $=\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ [1]

15. Let a, b, c be the d.r's of the required line. Cartesian equation of the line through the point $(1,2,-4)$ is

$\dfrac{x-x_1}{a}=\dfrac{y-y_1}{b}=\dfrac{z-z_1}{c}$

$\dfrac{x-1}{a}=\dfrac{y-2}{b}=\dfrac{z+4}{c}$ ... (i) [1]

Line (i) is perpendicular to the given two lines

$\therefore 3a-16b+7c=0$

Using $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$3a+8b-5c=0$

$$\frac{a}{80-56}=\frac{-b}{-15-21}=\frac{c}{24+48}=p \quad [1]$$

$\therefore\ a=24p, b=36p, c=72p$

or $a=2, b=3, c=6$

Putting the value of *a, b* and *c* in (i),

$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}$ is the required cartesian equation.

Point $(1,2-4)$ $\quad \therefore\ \vec{a}=\hat{i}+2\hat{j}-4\hat{k}$

D.r's 2, 3, 6 $\quad \therefore\ \vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$

Vector equation of the line

$\vec{r}=\vec{a}+\lambda\vec{b}$

$\therefore\ \vec{r}=\left(\hat{i}+2\hat{j}-4\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$ [1]

Cartesian equation :

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6} \quad [1]$$

16. LHS = $\left[\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}\right]$

$=\left(\vec{a}+\vec{b}\right).\left(\left(\vec{b}+\vec{c}\right)\times\left(\vec{c}+\vec{a}\right)\right)$ [1]

$=\left(\vec{a}+\vec{b}\right).\left(\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}\right)$

$=\left(\vec{a}+\vec{b}\right).\left(\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}\right)$ $\quad \left[\because \vec{c}\times\vec{c}=0\right]$ [1]

$=\vec{a}.\left(\vec{b}\times\vec{c}\right)+\vec{a}.\left(\vec{b}\times\vec{a}\right)+\vec{a}.\left(\vec{c}\times\vec{a}\right)+\vec{b}.\left(\vec{b}\times\vec{c}\right)$

$\quad +\vec{b}.\left(\vec{b}\times\vec{a}\right)+\vec{b}.\left(\vec{c}\times\vec{a}\right)$ [1]

$=\left[\vec{a},\vec{b},\vec{c}\right]+\vec{0}+\vec{0}+\vec{0}+\vec{0}+\left[\vec{a},\vec{b},\vec{c}\right]$

$=2\left[\vec{a},\vec{b},\vec{c}\right]$ = R.H.S. [1]

17. $\vec{a}+\vec{b}+\vec{c}=0$,

$\vec{a}+\vec{b}=-\vec{c}$ [1]

Squaring both the sides

$\left(\vec{a}+\vec{b}\right)\left(\vec{a}+\vec{b}\right)=\left(-\vec{c}\right)\left(-\vec{c}\right)$

$\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b}=\left|\vec{c}\right|^2$

$\left|\vec{a}\right|^2+2\vec{a}.\vec{b}+\left|\vec{b}\right|^2=\left|\vec{c}\right|^2$ [1]

$\left|\vec{a}\right|^2+2\left|\vec{a}\right|.\left|\vec{b}\right|\cos\theta+\left|\vec{b}\right|^2=\left|\vec{c}\right|^2$

$(3)^2+2(3)(5)\cos\theta+(5)^2=(7)^2$

$9+30\cos\theta+25=49$

$\Rightarrow 30\cos\theta=49-34=15$ [1]

$\Rightarrow \cos\theta=\frac{15}{30}=\frac{1}{2}$

$\therefore \cos\theta=\cos 60^o$

$\Rightarrow \theta=60^o$, Hence proved. [1]

18. $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

Squaring both the sides

$\left(\vec{a}+\vec{b}+\vec{c}\right)\left(\vec{a}+\vec{b}+\vec{c}\right)=\vec{0}$

Using $\vec{a}.\vec{a}=\vec{a}^2=\left|\vec{a}\right|^2, \vec{b}.\vec{a}=\vec{a}.\vec{b}$ [1]

$\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{a}.\vec{c}+\vec{b}.\vec{a}+\vec{b}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}+\vec{c}.\vec{b}+\vec{c}.\vec{c}=\vec{0}$

$a^2+\vec{a}.\vec{b}+\vec{c}.\vec{a}+\vec{a}.\vec{b}+\vec{b}^2+\vec{b}.\vec{c}+\vec{c}.\vec{a}+\vec{b}.\vec{c}+\vec{c}^2=\vec{0}$

$\left|\vec{a}\right|^2+\left|\vec{b}\right|^2+\left|\vec{c}\right|^2+2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)=\vec{0}$ [1]

$(5)^2+(12)^2+(13)^2+2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)=\vec{0}$

$25+144+169+2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)=\vec{0}$ [1]

$2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)=-338$

$\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}=\frac{-338}{2}=-169$ [1]

19. Given : $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$

$\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$

$\vec{a}+\vec{b}=4\hat{i}+4\hat{j}+0\hat{k}$

$\vec{a}-\vec{b}=2\hat{i}+0\hat{j}+4\hat{k}$ [1]

A vector perpendicular to

$\vec{a}+\vec{b}\ and\ \vec{a}-\hat{b}\ is\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)$

Let $\vec{c}=\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)$

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 4 & o\\ 2 & 0 & 4\end{vmatrix}$$ [1]

$|\vec{c}|=\hat{i}(16-0)-\hat{j}(16-0)+\hat{k}(0-8)$

$=16\hat{i}-16\hat{j}-8\hat{k}$

$=\sqrt{256+256+64}=\sqrt{576}=24$ [1]

Required vector $=\pm\dfrac{\vec{c}}{|\vec{c}|}$

$=\pm\dfrac{16\hat{i}-16\hat{j}-8\hat{k}}{24}$

$=\pm\dfrac{2}{3}\hat{i}\mp\dfrac{2}{3}\hat{j}\mp\dfrac{1}{3}\hat{k}$ [1]

20. Given: $L_1:\dfrac{-x+2}{-2}=\dfrac{y-1}{7}=\dfrac{z+3}{-3}$

$\dfrac{(x-2)}{2}=\dfrac{y-1}{7}=\dfrac{z+3}{-3}$ [1]

Direction ratios of line $(1)\,are\;2,7-3$

$L_2:\dfrac{x+2}{-1}=\dfrac{2(y-4)}{4}=\dfrac{z-5}{4}$ [1]

Direction ratios of line $(2)\,are-1,2,4$

Let θ be the required angle.

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$=\dfrac{2(-1)+7(2)+(-3)(4)}{\sqrt{4+49+9\sqrt{1+4+16}}}$ [1]

$=\dfrac{-2+14-12}{\sqrt{62}\sqrt{21}}=0$

$\cos\theta=0 \quad\Rightarrow\cos\theta=\cos 90^\circ$

$\theta=90^\circ\; or\dfrac{\pi}{2}.$ [1]

21. Given $\vec{a}=4\hat{i}+5\hat{j}-\hat{k},\quad \vec{b}=\hat{i}-4\hat{j}+5\hat{k}$ and

$\vec{c}=3\hat{i}+\hat{j}-\hat{k}\quad \vec{d}\cdot\vec{a}=21,\ \vec{d}\perp\vec{c}$ Also, $\vec{d}\perp\vec{b}$, and .

$\Rightarrow\vec{d}=\mu\left(\vec{c}\perp\vec{b}\right)$ [1]

$$\Rightarrow\vec{d}=\mu\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & -1\\ 1 & -4 & 5\end{vmatrix}$$ [1]

$\Rightarrow\vec{d}=\mu\left[\hat{i}(5-4)-\hat{j}(15+1)+\hat{k}(-12-1)\right]$

$\Rightarrow\vec{d}=\mu\left[\hat{i}-16\hat{j}-13\hat{k}\right]$ (1)

From the given equation we have,

$\vec{d}\cdot\vec{a}=21$

$\mu\left[\hat{i}-16\hat{j}-13\hat{k}\right]\left(4\hat{i}+5\hat{j}-\hat{k}\right)=21$

$\mu(4-80+13)=21$

$-63\mu=21$

$\mu=\dfrac{-1}{3}$ [1]

Substituting the values of in (1), we get

$\vec{d}=\dfrac{-1}{3}\left[\hat{i}-16\hat{j}-13\hat{k}\right]$

$\vec{d}=\dfrac{-1}{3}\hat{i}+\dfrac{16}{3}\hat{j}+\dfrac{13}{3}\hat{k}$ [1]

22. $\vec{a}=\dfrac{1}{7}\left(2\hat{i}+3\hat{j}+6\hat{k}\right)=\left(\dfrac{2}{7}\hat{i}+\dfrac{3}{7}\hat{j}+\dfrac{6}{7}\hat{k}\right)$

$|\vec{a}|=\sqrt{\left(\dfrac{2}{7}\right)^2+\left(\dfrac{3}{7}\right)^2+\left(\dfrac{6}{7}\right)^2}$ [1]

$=\sqrt{\dfrac{4}{49}+\dfrac{9}{49}+\dfrac{36}{49}}=\sqrt{\dfrac{49}{49}}=1$

$\vec{b}=\dfrac{1}{7}\left(3\hat{i}-6\hat{j}+2\hat{k}\right)=\left(\dfrac{3}{7}\hat{i}-\dfrac{6}{7}\hat{j}+\dfrac{2}{7}\hat{k}\right)$

$\left|\vec{b}\right|=\sqrt{\left(\dfrac{3}{7}\right)^2+\left(\dfrac{-6}{7}\right)^2+\left(\dfrac{2}{7}\right)^2}$ [1]

$=\sqrt{\dfrac{9}{49}+\dfrac{36}{49}+\dfrac{4}{49}}$

$=\sqrt{\dfrac{49}{49}}=1$

$\vec{c}=\frac{1}{7}\left(6\hat{i}+2\hat{j}-3\hat{k}\right)=\left(\frac{6}{7}\hat{i}+\frac{2}{7}\hat{j}-\frac{3}{7}\hat{k}\right)$

$|\vec{c}|=\sqrt{\left(\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2+\left(\frac{-3}{7}\right)^2}$

$=\sqrt{\frac{36}{49}+\frac{4}{49}+\frac{9}{49}}=\sqrt{\frac{49}{49}}=1$ [1]

Thus the given three vectors are unit vectors

Now,

$\vec{a}.\vec{b}=\frac{2}{7}\times\frac{3}{7}+\frac{3}{7}\times-\frac{6}{7}+\frac{6}{7}\times\frac{2}{7}=\frac{6}{49}-\frac{18}{49}+\frac{12}{49}=0$

$\vec{b}.\vec{c}=\frac{3}{7}\times\frac{6}{7}+\frac{-6}{7}\times\frac{2}{7}+\frac{2}{7}\times\frac{-3}{7}=\frac{18}{49}-1249-\frac{6}{49}=0$

$\vec{c}\vec{a}=\frac{6}{7}\times\frac{2}{7}+\frac{2}{7}\times\frac{3}{7}+\frac{-3}{7}\times\frac{6}{7}=\frac{12}{49}+\frac{6}{49}-\frac{18}{49}=0$

Hence the given three vectors are mutually perpendicular to each other. [1]

23. $\vec{a}=\hat{i}-\hat{j}+7\hat{k}$

$\vec{b}=5\hat{i}-\hat{j}+\lambda\hat{k}$

$\vec{a}+\vec{b}=6\hat{i}-2\hat{j}+(7+\lambda)\hat{k}$ [1]

$\vec{a}-\vec{b}=-4\hat{i}-0\hat{j}+(7-\lambda)\hat{k}$ [1]

Given $\left(\vec{a}+\vec{b}\right)and\left(\vec{a}-\vec{b}\right)$ are perpendicular

$\therefore\left(\vec{a}+\vec{b}\right)\left(\vec{a}-\vec{b}\right)=0$

$\left\{6\hat{i}-2\hat{j}+(7+\lambda)\hat{k}\right\}\times\left\{-4\hat{i}-0\hat{j}+(7-\lambda)\hat{k}\right\}=0$

$6(-4)+0(-2)+(7+\lambda)(7-\lambda)=0$ [1]

$-24+49-\lambda^2=0$

$\lambda^2=25\Rightarrow\lambda=\pm5$ [1]

24. If the given lines are intersecting then the shortest distance between the lines is zero and also they have a common point

$\vec{r}=3\hat{i}+2\hat{j}-4\hat{k}+\lambda\left(\hat{i}+2\hat{j}+2\hat{k}\right)$

$\Rightarrow\frac{x-3}{1}=\frac{y-2}{2}=\frac{z+4}{2}(=\lambda)(\text{Let})$ [1]

Let P is $(\lambda+3,2\lambda+2,2\lambda-4)$

Also, $\vec{r}=5\hat{i}-2\hat{j}+\mu\left(3\hat{i}+2\hat{j}+6\hat{k}\right)$

$\Rightarrow\frac{x-5}{3}=\frac{y+2}{2}=\frac{z-0}{6}(=\mu)(\text{Let})$ [1]

Let Q is $(3\mu+5,2\mu-2,6\mu)$

If lines are intersecting then P and Q will be same.

$\lambda+3=3\mu+5$ ....(1)

$2\lambda+2=2\mu-2$ ....(2)

$2\lambda-4=6\mu$ ....(3)

Subtracting (3) from (2)

$2\lambda+2=2\mu-2$
$2\lambda-4=6\mu$
$-\quad+\quad-$

$6=-4\mu-2$ [1]

Put $\mu=-2$ in (3)

$2\lambda-4=6(-2)$
$2\lambda=-12+4$
$2\lambda=-8$
$\lambda=-4$

Put μ and λ in (1)

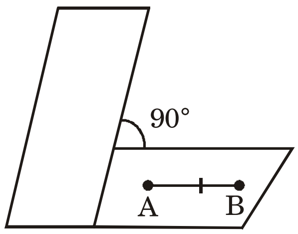
$\lambda+3=3\mu+5$
$-4+3=3(-2)+5$
$-1=-1$

$\therefore$ from $\lambda=-4$ then P is $(-1,-6,-12)$
from $\lambda=-2$ then Q is $(-1,-6,-12)$

as P and Q are same.

$\therefore$ lines are intersecting lines and their point of intersection is $(-1,-6,-12)$. [1]

25.



$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$\overrightarrow{AB}=-3\hat{i}+2\hat{j}+5\hat{k}$

Given plane $x-2y+4z=10$

$\therefore\quad\overrightarrow{n_1}=\hat{i}-2\hat{j}+4\hat{k}$ [1]

The required plane is perpendicular to given plane. Therefore $\vec{n}$ of required plane will be perpendicular to $\overrightarrow{n_1}$ and $\overrightarrow{AB}$.

$\therefore \vec{n} \parallel \left(\overrightarrow{n_1} \times \overrightarrow{AB}\right)$

$\overrightarrow{n_1} = \hat{i} - 2\hat{j} + 4\hat{k}$ [1]

$\overrightarrow{AB} = -3\hat{i} + 2\hat{j} + 5\hat{k}$

$\therefore \overrightarrow{n_1} \times \overrightarrow{AB} = -18\hat{i} - 17\hat{j} - 4\hat{k}$

$\therefore$ required plane is

$\vec{r}.\vec{n} = \vec{a}.\vec{n}$

$\vec{r}.\left(-18\hat{i} - 17\hat{j} - 4\hat{k}\right) = \left(2\hat{i} + \hat{j} - \hat{k}\right).\left(-18\hat{i} - 17\hat{j} - 4\hat{k}\right)$ [1]

$\vec{r}.\left(-18\hat{i} - 17\hat{j} - 4\hat{k}\right) = -36 - 17 + 4$

$\vec{r}.\left(-18\hat{i} - 17\hat{j} - 4\hat{k}\right) = -49$

$18x + 17y + 4z = 49$ [1]

26. Line L is passing through point = $\left(2\hat{i} - \hat{j} - 3\hat{k}\right)$

If $L_1 \Rightarrow \vec{r} = \left(\hat{i} + \hat{j} - \hat{k}\right) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$

$L_2 \Rightarrow \vec{r} = \left(2\hat{i} - \hat{j} - 3\hat{k}\right) + \mu\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$ [1]

given that line L is perpendicular to L_1 and L_2

Let dr of line L = a_1, a_2, a_3

The equation of L in vector form

$\Rightarrow$ k is any constant.

so by condition that L_2 is perpendicular to L.

$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

$2a_1 - 2a_2 + a_3 = 0$ ....(1) [1]

And also $L \perp L_2$

so, $a_1 + 2a_2 + 2a_3 = 0$ ....(2)

Solve (1),(2)

$3a_1 + 3a_3 = 0$ [1]

$\Rightarrow a_3 = -a_1$

Put it in (2)

$a_1 + 2a_2 - 2a_1 = 0$

$a_2 = \frac{a_1}{2}$ *let*

So D.R. of L = $\left(a_1, \frac{a_1}{2}, -a_1\right)$

So we can say dr of L= $\left(1, \frac{1}{2}, -1\right)$

So equation of L in vector form:

$\vec{r} = \left(2\hat{i} - \hat{j} - 3\hat{k}\right) + k\left(\hat{i} + \frac{\hat{j}}{2} - \hat{k}\right)$

3-D form $\rightarrow \frac{x-2}{1} = \frac{y+1}{\frac{1}{2}}$

$= \frac{z-3}{-1}$ [1]

27. Here $\overrightarrow{AB} = -4\hat{i} - 6\hat{j} - 2\hat{k}$

$\overrightarrow{AC} = -\hat{i} + 4\hat{j} + 3\hat{k}$ [1]

$\overrightarrow{AD} = -8\hat{i} - \hat{j} + 3\hat{k}$ [1]

For them to be coplanar, $[\overrightarrow{AB} \quad \overrightarrow{AC} \quad \overrightarrow{AD}] = 0$

i.e. $\begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = -60 + 126 - 66 = 0$ [1]

$\therefore$ Points A, B, C and D are co – planar [1]

28. Now, vectors

$\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}$ *represents the sides of* ΔABC

$\therefore \overrightarrow{AB} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k}$

$= -\hat{i} - 2\hat{j} - 6\hat{k}$

$\Rightarrow \left|\overrightarrow{AB}\right| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2}$

$= \sqrt{1 + 4 + 36} = \sqrt{41}$ [1]

$\overrightarrow{BC} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$

$\Rightarrow \left|\overrightarrow{BC}\right| = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{4+1+1} = \sqrt{6}$

$\overrightarrow{CA} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k}$

$= -\hat{i} + 3\hat{j} + 5\hat{k}$ [1]

$\Rightarrow \left|\overrightarrow{CA}\right| = \sqrt{(-1)^2+(3)^2+(5)^2} = \sqrt{1+9+25} = \sqrt{35}$

$\therefore \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{CA}\right|^2 = 6+35 = 41 = \left|\overrightarrow{AB}\right|^2$ [1]

Hence, ΔABC *is a right angled triangle.*

Area of $\Delta ABC = \frac{1}{2} X\ Base\ X\ Height$

$= \frac{1}{2} X \left|\overrightarrow{BC}\right| X \left|\overrightarrow{CA}\right|$

$= \frac{1}{2} X 6 X 35$

$= 105$ *square units.* [1]

29. We have

P. V of A = $3\hat{i}+6\hat{j}+9\hat{k}$ [1]

P. V. of B = $\hat{i}+2\hat{j}+3\hat{k}$ [1]

$\overrightarrow{AB} = -2\hat{i}-4\hat{j}-6\hat{k}$

$\overrightarrow{AC} = -\hat{i}-3\hat{j}-8\hat{k}$

$\overrightarrow{AD} = \hat{i}+(\lambda-9)\hat{k}$

Now,

$\overrightarrow{AB}.\left(\overrightarrow{AC}\times\overrightarrow{AD}\right) \Rightarrow \begin{vmatrix} -2 & -4 & -6 \\ -1 & -3 & -8 \\ 1 & 0 & (\lambda-9) \end{vmatrix} = 0$ [1]

$\Rightarrow -2(-3\lambda+27)+4(-\lambda+9+8)-6(0+3) = 0$

$\Rightarrow 6\lambda-54-4\lambda+68-18 = 0$

$2\lambda-4 = 0$

$\lambda = 2$

$\therefore \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar and so the points A, B, C and D are coplanar. [1]

30. Let the given vectors are $\vec{a} = \hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b} = \widehat{3i}-2\hat{j}+\hat{k}$

$|\vec{a}| = \sqrt{1^2+(-2)^2+3^2} = \sqrt{1+4+9} = \sqrt{14}$

$|\vec{b}| = \sqrt{3^2+(-2)^2+1^2} = \sqrt{9+4+1} = \sqrt{14}$ [1]

Now, $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 3 & -2 & 1 \end{vmatrix}$ [1]

$= \hat{i}(-2+6)-\hat{j}(1-9)+\hat{k}(-2+6)$

$= 4\hat{i}+8\hat{j}+4\hat{k}$ [1]

$\left|\vec{a}\times\vec{b}\right| = \sqrt{4^2+8^2+4^2}$

$= \sqrt{16+64+16}$

$= \sqrt{96} = 4\sqrt{6}$ [1]

Also, we know that $\sin\theta = \frac{\left|\vec{a}\times\vec{b}\right|}{|\vec{a}||\vec{b}|}$ [1]

$\sin\theta = \frac{4\sqrt{6}}{\sqrt{14}\sqrt{14}}$

$= \frac{4\sqrt{6}}{14}$

$= \frac{2\sqrt{6}}{7}$ [1]

Smart Notes ....

Smart Notes ....

CHAPTER 11

Three Dimensional Geometry

Chapter Analysis of Previous Three Years' Board Exams

Number of Questions Asked in Exams

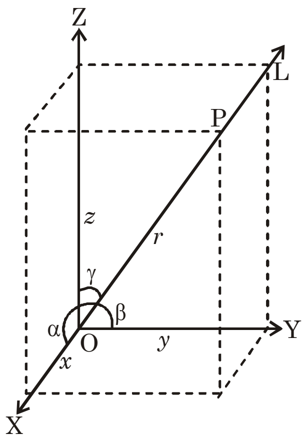
Topics	2018		2017		2016	
	Delhi	**All India**	**Delhi**	**All India**	**Delhi**	**All India**
Equation of Plane			6 marks		1 marks	1 marks
Vector and Cartesian Equation of Line					4 marks	
Distance between Plane	6 marks	6 marks		1, 2 marks		
Co-ordiante of Points			1 marks	6 marks		4 marks
Image of a point in a Plane						6 marks
Section Formula				6 marks		
Shortest distance between lines	4 marks	4 marks				

[TOPIC 1] Direction Cosines and Lines

Summary

- **Direction Cosines:**

  These are the cosines of the angles made by the line with the positive directions of the coordinate axes.



The direction cosines of line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are

$$\pm\frac{x_2-x_1}{AB}, \pm\frac{y_2-y_1}{AB}, \pm\frac{z_2-z_1}{AB} \text{ where}$$

$$AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Assume the direction cosines of the line are l, m and n and that the line is passing through the point $A(x_1, y_1, z_1)$, then the equation of the line is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

Also, $l^2 + m^2 + n^2 = 1$

- **Direction Cosines:**

  Direction ratios are any three numbers which are proportional to the direction cosines.

  Let the direction ratios be a, b and c, then

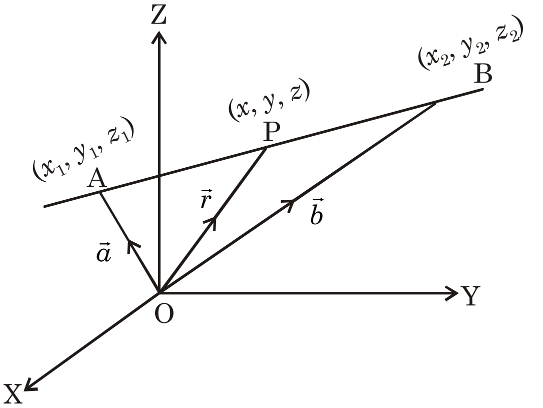
$$l=\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m=\pm\frac{b}{\sqrt{a^2+b^2+c^2}} \text{ and}$$

$$n=\pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

- **Lines:**

  ➤ Equation of the line which passes through two given points:

  Assume that the position vectors of $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are $\vec{a}$ and $\vec{b}$ respectively.



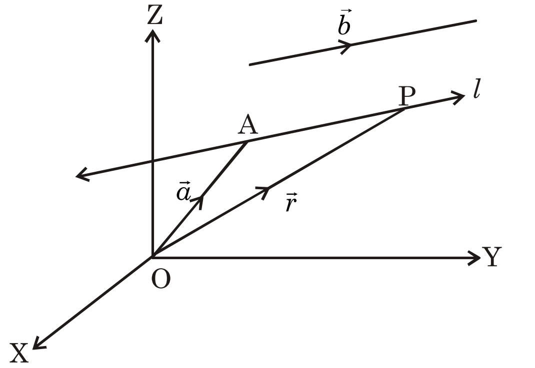
Then, the vector equation of the line is

$$\vec{r}=\vec{a}+k\left(\vec{b}-\vec{a}\right), k \in \mathbb{R}$$

The equation of the line in Cartesian form is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

➤ Equation of the line which passes through a given point having a given direction



The vector equation of the line is $\vec{r}=\vec{a}+\lambda\vec{b}$

The equation of the line in Cartesian form when it passes through $A(x_1, y_1, z_1)$ is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$ where the direction ratios are a, b and c.

The angle between the lines $\vec{r} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{r} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ is given by

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

The shortest distance between the lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ is given by

$$\left|\frac{\left(\vec{b_1} \times \vec{b_2}\right) \cdot \left(\vec{a_2} - \vec{a_1}\right)}{\left|\vec{b_1} \times \vec{b_2}\right|}\right|$$

The shortest distance between the lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is given by

$$\left|\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}\right|$$

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

1 Mark Questions

1. If the Cartesian equations of a line are $\frac{3 - x}{5} = \frac{y + 4}{7} = \frac{2z - 6}{4}$, Write the vector equation for the line.

   [ALL INDIA 2014]

2. If a line has direction ratios 2, -1, -2, then what are its direction cosines?

   [ALL INDIA 2018]

3. Write the direction cosines of the vector $-2\hat{i} + \hat{j} - 5\hat{k}$.

   [DELHI 2011]

4. What are the direction cosines of a line, which makes equal angles with the coordinate axes?

   [ALL INDIA 2014]

2 Marks Question

5. The x-coordinate of a point lies on the line joining the points $P(2, 2, 1)$ and $Q(5, 1, -2)$ is 4. Find its z-coordinate.

   [ALL INDIA 2017]

4 Marks Questions

6. Show that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ intersect. Also find their point of intersection.

   [DELHI 2014]

7. Find the shortest distance between the lines:

   $\vec{r} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(\hat{i} - 3\hat{j} + 2\hat{k}\right)$

   $\vec{r} = \left(4\hat{i} + 5\hat{j} + 6\hat{k}\right) + \mu\left(2\hat{i} + 3\hat{j} + \hat{k}\right)$

   [ALL INDIA 2011]

8. The scalar product of the vector $\vec{a} = \left(\hat{i} + \hat{j} + \hat{k}\right)$ with a unit vector along the sum of vector $\vec{b} = \left(2\hat{i} + 4\hat{j} - 5\hat{k}\right)$ and $\vec{c} = \left(\lambda\hat{i} + 2\hat{j} + 3\hat{k}\right)$ is equal to one. Find the value of λ and hence find the unit vector along $\vec{b} + \vec{c}$.

   [ALL INDIA 2014]

9. Find the coordinates of the foot of perpendicular drawn from the point $A(-1, 8, 4)$ to the line joining the points $B(0, -1, 3)$ and $C(2, -3, -1)$. Hence find the image of the point A in the line BC.

   [ALL INDIA 2016]

10. Find the shortest distance between the lines $\vec{r}=\left(4\hat{i}-\hat{j}\right)+\lambda\left(\hat{i}+2\hat{j}-3\hat{k}\right)$ and

$\vec{r}=\left(\hat{i}-\hat{j}+2\hat{k}\right)+\mu\left(2\hat{i}+4\hat{j}-5\hat{k}\right)$

[DELHI 2018]

11. Find the Vector and Cartesian equations of the line passing through the point (1, 2, –4) and perpendicular to the two lines

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7} \text{ and } \frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}.$$

[DELHI 2012]

6 Marks Questions

12. Find the vector and Cartesian equations of the line passing through the points (1,2,–4)and perpendicular to the two lines

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7} \text{and} \frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}.$$

[DELHI 2011]

13. Find the vector and Cartesian equations of a line passing through $(1,2,-4)$ and perpendicular to the two lines $\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$.

[ALL INDIA 2012, 2017]

Solutions

1. Given cartesian equations of a line is

$$\frac{3-x}{5}=\frac{y+4}{7}=\frac{2z-6}{4},$$

We can write it as

$$\frac{x-3}{-5}=\frac{y+4}{7}=\frac{z-3}{2}$$

So vector equation is

$$\vec{r}=\left(3\hat{i}-4\hat{j}+3\hat{k}\right)+\lambda\left(-5\hat{i}+7\hat{j}+2\hat{k}\right)$$ [1]

where λ is a constant.

2. Here direction ratios of line are 2, –1, –2

∴ Direction cosines of line are

$$\frac{2}{\sqrt{2^2+(-1)^2+(-2)^2}},\frac{-1}{\sqrt{2^2+(-1)^2+(-2)^2}},$$

$$\frac{-2}{\sqrt{2^2+(-1)^2+(-2)^2}} \text{ i.e., } \frac{2}{3},-\frac{1}{3},-\frac{2}{3}$$

3. $|\vec{r}|=\sqrt{(-2)^2+(1)^2+(-5)^2}$

$=\sqrt{4+1+25}=\sqrt{30}$

Let a_1, b_1, c_1 be the direction ratio of $\vec{r}$

Direction cosines of $\vec{r}$ are

$$\iota=\frac{a_1}{|\vec{r}|}, m=\frac{b_1}{|\vec{r}|}, n=\frac{c_1}{|\vec{r}|}$$

$$\iota=\frac{-2}{\sqrt{30}}, m=\frac{1}{\sqrt{30}}, n=\frac{-5}{\sqrt{30}}$$ [1]

4. Let direction cosines of line make angle α.

$l=\cos\alpha$

$m=\cos\alpha$

$n=\cos\alpha$

$l^2+m^2+n^2=1$

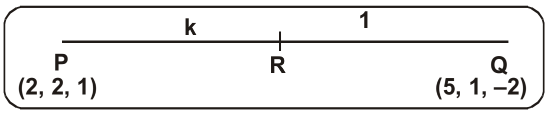
$\cos^2\alpha+\cos^2\alpha+\cos^2\alpha=1$ [½]

$3\cos^2\alpha=1$

$\cos^2\alpha=\frac{1}{3}$

$\cos\alpha=\pm\frac{1}{\sqrt{3}}$ [½]

5.



Let R divides PQ in the ratio $k:1$ [1]

$$R\left(\frac{5k+2}{k+1},\frac{k+2}{k+1},\frac{-2k+1}{k+1}\right)$$

given x co-ordinate of $R=4$

$\therefore \frac{5k+2}{k+1}=4$

$\Rightarrow k=2$

$\therefore$ z co-ordinate $=\frac{-2(2)+1}{2+1}=-1$ [1]

6. Given lines are $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}$ and $\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}$

Let $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}=p$ ...(i) [1]

$x=3p-1, y=5p-3, z=7p-5$

So, the coordinates of a general point on line (i) are

$(3p-1, 5p-3, 7p-5)$ ...(ii)

Also, let $\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}=q$

$x=q+2, y=3q+4, z=5q+6$

So, the coordinates of a general point on line (iii) are $(q+2, 3q+4, 5q+6)$

If the line intersect, then they have a common point.

So, for some values of p and q, we must have

$3p-1=q+2$

$\Rightarrow 3p-q=3$

$3p-3=q$ ...(iv) [1]

$7p-5=5q+6$

$\Rightarrow 7p-5q=11$ ...(v)

$5p-3=3q+4$

$\Rightarrow 5p-3q=7$

$\Rightarrow 5p-3(3p-3)=7$ (from equation (iv))

$\Rightarrow 5p-9p+9=7$

$\Rightarrow 5p-9p+9=7$

$\Rightarrow -4p=-2$ ...(vi)

From (iv) and (vi), $q=\frac{3}{2}-3=-\frac{3}{2}$

Putting the value of p and q in (v), [1]

$7\left(\frac{1}{2}\right)-5\left(\frac{-3}{2}\right)=11$

$\frac{7+15}{2}=11$

11 = 11, which is true.

Since $p=\frac{1}{2}, q=\frac{-3}{2}$ satisfy equation (v)

So, the given lines intersect.

Putting the value of p in (ii), we get

$\left(\frac{3}{2}-1, \frac{5}{2}-3, \frac{7}{2}-5\right)$

$\therefore \left(\frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}\right)$ is the required point of intersection of given lines. [1]

7. $\vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(\hat{i}-3\hat{j}+2\hat{k}\right)$

and $\vec{r}=\left(4\hat{i}+5\hat{j}+6\hat{k}\right)+\mu\left(2\hat{i}+3\hat{j}+\hat{k}\right)$

$\vec{a_1}=\hat{i}+2\hat{j}+3\hat{k}$

$\vec{b_1}=\hat{i}-3\hat{j}+2\hat{k}$

$\vec{a_2}=4\hat{i}+5\hat{j}+6\hat{k}$

$\vec{b_2}=2\hat{i}+3\hat{j}+\hat{k}$ [1]

Shortest distance between the above lines,

$$d=\left|\frac{\left(\vec{b_1}\times\vec{b_2}\right).\left(\vec{a_2}-\vec{a_1}\right)}{\left|\vec{b_1}\times\vec{b_2}\right|}\right|$$

$$\vec{a_2}-\vec{a_1}=\left(4\hat{i}+5\hat{j}+6\hat{k}\right)-\left(\hat{i}+2\hat{j}+3\hat{k}\right)$$
$$=3\hat{i}+3\hat{j}+3\hat{k}$$

$$\vec{b_1}\times\vec{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 2\\ 2 & 3 & 1\end{vmatrix}=-9\hat{i}+3\hat{j}+9\hat{k}$$

$\left|\vec{b_1}\times\vec{b_2}\right|=\sqrt{(-9)^2+3^2+9^2}=\sqrt{19}$ [1]

Now,

$$d=\left|\frac{\left(\vec{b_1}\times\vec{b_2}\right).\left(\vec{a_2}-\vec{a_1}\right)}{\left|\vec{b_1}\times\vec{b_2}\right|}\right|$$ [1]

$$d=\left|\frac{-9\times3+3\times3+9\times3}{3\sqrt{19}}\right|$$

$$d=\left|\frac{-27+9+27}{3\sqrt{19}}\right|$$

$$d=\frac{3}{\sqrt{19}}$$ [1]

8. Given $\vec{a} = \left(\hat{i} + \hat{j} + \hat{k}\right)$

$\vec{b} = \left(2\hat{i} + 4\hat{j} - 5\hat{k}\right)$

$\vec{c} = \left(\lambda\hat{i} + 2\hat{j} + 3\hat{k}\right)$

So, $\vec{b} + \vec{c} = \left((2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}\right)$ [1]

Unit vector along

$$\vec{b} + \vec{c} = \frac{\left(2+\lambda\right)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda)^2 + 36 + 4}}$$

$$= \frac{\left(2+\lambda\right)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda)^2 + 40}}$$ [1]

given that dot product of $\vec{a}$ with the unit vector of $\vec{b} + \vec{c}$ is equal to 1. So, apply given condition

$$= \frac{(2+\lambda) + 6 - 2}{\sqrt{(2+\lambda)^2 + 40}} = 1$$

$\Rightarrow (2+\lambda) + 4 = \sqrt{(2+\lambda)^2 + 40}$ [1]

Squaring

$36 + \lambda^2 + 12\lambda = 4 + \lambda^2 + 4\lambda + 40$

$\Rightarrow 8\lambda = 8$

$\Rightarrow \lambda = 1$ [1]

9. Equation of BC

$$\frac{x-0}{2-0} = \frac{y+1}{-3+1} = \frac{z-3}{-1-3}$$

$$\Rightarrow \frac{x}{2} = \frac{y+1}{-2} = \frac{z-3}{-4}$$

$\dfrac{x}{1} = \dfrac{y+1}{-1} = \dfrac{z-3}{-2} = \lambda$ [1]

$\Rightarrow D(\lambda, -\lambda - 1, -2\lambda + 3)$

DR's of AD

$\Rightarrow \lambda + 1, -\lambda - 9, -2\lambda + 3 - 4$

$\Rightarrow \lambda + 1, -\lambda - 9, -2\lambda - 1$

$\therefore$ AD $\perp$ BC [1]

$\Rightarrow (\lambda + 1)(1) + (-\lambda - 9)(-1) + (-2\lambda - 1)(-2) = 0$

$\Rightarrow \lambda + 1 + \lambda + 9 + 4\lambda + 2 = 0$

$\Rightarrow 6\lambda + 12 = 0$

$\Rightarrow \lambda = -2$ [1]

Foot of perpendicular to $D(-2, 1, 7)$

Let image of A w.r. to line BC is $E(x_1, y_1, z_1)$

Mid point of AE

$$\left(\frac{x_1 - 1}{2}, \frac{y_1 + 8}{2}, \frac{z_1 + 4}{2}\right) = (-2, 1, 7)$$

$x_1 - 1 = -4,\ y_1 + 8 = 2,\ z_1 + 4 = 14$

$(x_1 = -3)\ (y_1 = -6)\ (z_1 = 10)$

$E(-3, -6, 10)$ [1]

10. Shortest distance between the lines, $\vec{r} = \overrightarrow{a_1} + \lambda\overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \mu\overrightarrow{b_2}$ is given by ,

$$d = \left|\frac{\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right)\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right)}{\left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right|}\right|$$ [1]

Comparing the given equations we get,

$\overrightarrow{a_1} = \left(4\hat{i} - \hat{j}\right)$

$\overrightarrow{b_1} = \left(\hat{i} + 2\hat{j} - 3\hat{k}\right)$

$\overrightarrow{a_2} = \left(\hat{i} - \hat{j} + 2\hat{k}\right)$

$\overrightarrow{b_2} = \left(2\hat{i} + 4\hat{j} - 5\hat{k}\right)$

$\overrightarrow{a_2} - \overrightarrow{a_1} = \left(-3\hat{i} + 2\hat{k}\right)$

$$\overrightarrow{b_1} \times \overrightarrow{b_2} = \begin{vmatrix} i & j & k \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix}$$ [1]

$= \hat{i}(-10 + 12) - \hat{j}(-5 + 6) + \hat{k}(4 - 4)$

$= 2\hat{i} - \hat{j}$

$\Rightarrow \left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right| = \sqrt{4 + 1} = \sqrt{5}$ [1]

Substituting the values in the shortest distance formula we get

$$d = \left|\frac{(2\hat{i} - \hat{j}).(-3\hat{i} + 2\hat{k})}{\sqrt{5}}\right|$$

$$= \left|\frac{-6+0+0}{\sqrt{5}}\right|$$

$$= \frac{6}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{6\sqrt{5}}{5} \text{ units} \quad [1]$$

11. Let the Cartesian equation of line passing through (1, 2, –4) be

$$\frac{x-1}{a} = \frac{y-2}{b} = \frac{z+4}{c} \quad \dots(1)$$

Given lines are

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \quad \dots(2)$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \quad \dots(3) \quad [1]$$

Obviously parallel vectors of $\vec{b}_1, \vec{b}_2$ and $\vec{b}_3$ of (1), (2) and (3) respectively are

$$\vec{b}_1 = a\hat{i} + b\hat{j} + c\hat{k}$$

$$\vec{b}_2 = 3\hat{i} - 16\hat{j} + 7\hat{k}$$

$$\vec{b}_3 = 3\hat{i} + 8\hat{j} - 5\hat{k} \quad [1]$$

From equation

$$(1) \perp (2)$$

$$\Rightarrow \vec{b}_1 \perp \vec{b}_2$$

$$\Rightarrow \vec{b}_1.\vec{b}_2 = 0$$

$$(1) \perp (3)$$

$$\Rightarrow \vec{b}_1 \perp \vec{b}_3$$

$$\Rightarrow \vec{b}_1.\vec{b}_3 = 0$$

Hence, $3a - 16b + 7c = 0$ ... (4)

And $3a + 8b - 5c = 0$ ... (5)

From equations (4) and (5)

$$\frac{a}{80-56} = \frac{b}{21+15} = \frac{c}{24+48}$$

$$\frac{a}{24} = \frac{b}{36} = \frac{c}{72}$$

$$\Rightarrow \frac{a}{2} = \frac{b}{3} = \frac{c}{6} = \lambda \text{ let} \quad [1]$$

Putting the value of a, b, c in Eq. (i) we get required Cartesian equation of line as

$$\Rightarrow \frac{x-1}{2\lambda} = \frac{y-2}{3\lambda} = \frac{z+4}{6\lambda}$$

$$\Rightarrow \frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

Hence vector equation is

$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}) \quad [1]$$

12. Given a, b, c be the direction ratio of the required line Equation of required line passing through point $(1, 2 - 4)$ *is*

$$\frac{x-1}{a} = \frac{y-2}{b} = \frac{z+4}{c} \quad [1]$$

Given first line is,

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$$

Given second line is,

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \quad [1]$$

Required this is $\perp$ to given 1st line

$$3a - 16b + 7c = 0$$

Required line is $\perp$ to given 2nd line [1]

$$3a + 8b - 5c = 0$$

$$\frac{a}{80-56} = \frac{-b}{-15-21} = \frac{c}{24+48}$$

$$\frac{a}{24} = \frac{-b}{-36} = \frac{c}{72} = \rho(let) \quad [1]$$

$$a = 24\rho, \quad b = 36\rho, \quad c = 72\rho$$

$$or \ a = 2, \quad b = 3, \quad c = 6$$

Putting the values of a, b, c, in (i),we have Cartesian equation of a line

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}$$ [1]

Point $(1,2-4)$, DR's are $2,3,6$

$\vec{A}=\hat{i}+2\hat{j}-4\hat{j}$, $\vec{B}=2\hat{i}+3\hat{j}+6\hat{k}$

Vector equation of a line $\vec{r}=\vec{A}+\lambda\vec{B}$

$=\left(\hat{i}+2\hat{j}-4\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$ [1]

13. Let a, b, c be the direction ratio of the required line.

Equation of required line passing through $(1,2,-4)$ is

$$\frac{x-1}{a}=\frac{y-2}{b}=\frac{z+4}{c} \quad (1)$$

Given first line is,

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$$ [1]

Given second line is,

$$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$$ [1]

Required line is perpendicular to given first line.

$\therefore 3a-16b+7c=0$

Required line is perpendicular to given second line.

$\therefore 3a+8b-5c=0$ [1]

$$\frac{a}{80-56}=\frac{-b}{-15-21}=\frac{c}{24+48}$$

$$\frac{a}{24}=\frac{-b}{-36}=\frac{c}{72}=p \quad \text{(Let)}$$

$\therefore\ a=24p,\ b=36p,\ c=72p$ [1]

$a=2, b=3,\ c=6$ Dividing by

Putting the values of a, b, c in (1), we have

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}$$

Point $(1,2,-4)$,D.rs are $2,3,6$

$\therefore\ \overline{A}=\hat{i}+2\hat{j}-4\hat{k},$

$\overline{B}=2\hat{i}+3\hat{j}+6\hat{k}$ [1]

Vector equation of a line, $\vec{r}=\overline{A}+\lambda\overline{B}$

$=\left(\hat{i}+2\hat{j}-4\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$ [1]

[TOPIC 2] Plane

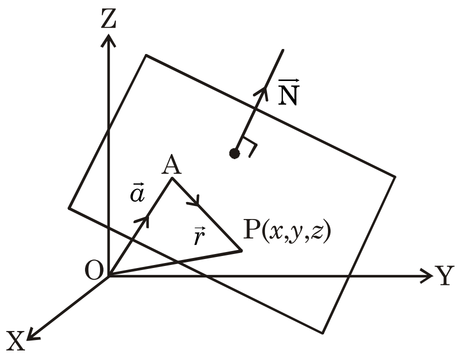
Summary

- **Plane**

  A surface so that when the two points are taken on it, the line segment lies joining the two points lies on the surface is called a plane.

  The equation of the plane is $\vec{r}\cdot\hat{n} = d$ where $\hat{n}$ is the unit vector normal to plane of origin. The equation of the plane in normal form is $lx + my + nz = d$ where l, m, n are direction cosines.

- **Equation of a plane perpendicular to a given vector**

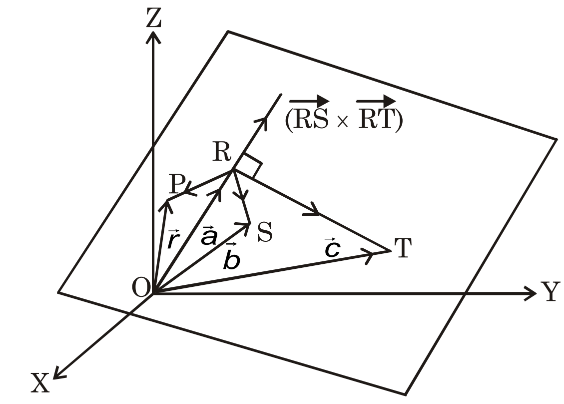


  The equation of a plane through a point whose position vector is $\vec{a}$ and perpendicular to the vector $\vec{N}$ is $\left(\vec{r}-\vec{a}\right)\cdot\vec{N} = 0$

  Equation of a plane perpendicular to a given vector and passing through a given point is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

- **Equation of a plane passing through three non collinear points**

  Let the non collinear points be $R(x_1, y_1, z_1)$, $S(x_1, y_2, z_2)$, $T(x_3, y_3, z_3)$ and $\vec{r}$ be the position vector.



  Equation of a plane passing through three non collinear points is

$$\left(\vec{r}-\vec{a}\right)\cdot\left[\left(\vec{b}-\vec{a}\right)\times\left(\vec{c}-\vec{a}\right)\right] = 0$$

  In Cartesian plane,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

  The equation of the plane in the intercept form is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$ where a, b, c are x, y, z intercepts respectively.

- **The plane passing through intersection of two given lines**

  It has the equation

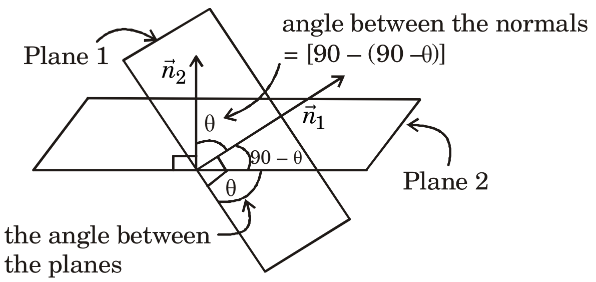
$$\vec{r}\cdot\left(\vec{n_1}+\lambda\vec{n_2}\right) = d_1 + \lambda d_2$$

  In Cartesian system,

$$(A_1x + B_1y + C_1z - d_1) + l(A_2x + B_2y + C_2z - d_2) = 0$$

- **Angle between two planes**

  The angle between the planes is given by the angle between their normals.



Let the angle between the planes be θ.

$$\cos\theta = \left|\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right|$$ where $\vec{n}_1, \vec{n}_2$ are normal to the planes.

In Cartesian form, $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$

$$\cos\theta = \left|\frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}\right|$$

The distance between a plane $Ax + By + Cz + D$ and the point (x_1, y_1, z_1) is given by

$$\left|\frac{A_1x + B_1y + C_1z + D_1}{\sqrt{A^2 + B^2 + C^2}}\right|$$

The angle between the line $\vec{r} = \vec{a} + \lambda\vec{b}$ and the plane $\vec{r} \cdot \hat{n} = d$ is $\sin\varphi = \left|\frac{\vec{b} \cdot \hat{n}}{|\vec{b}||\hat{n}|}\right|$

PREVIOUS YEARS' EXAMINATION QUESTIONS

TOPIC 2

1 Mark Questions

1. Find the distance between the planes $2x - y + 2z = 5$ and $5x - 2.5y + 5z = 20$.

   [ALL INDIA 2017]

2. Write the sum of intercepts cut off by the plane $\vec{r}.(2\hat{i} + \hat{j} - \hat{k}) - 5 = 0$ on the three axes.

   [ALL INDIA 2016]

3. Find the acute angle between the plane $5x - 4y + 7z - 13 = 0$ and the y-axis

   [ALL INDIA 2015]

4. Find the length of the perpendicular drawn from origin to the $2x - 3y + 6z + 21 = 0$ plane .

   [ALL INDIA 2013]

5. Find the vector equation of a plane which is at a distance of 5 units from the origin and its normal vector $2\hat{i} - 3\hat{j} + 6\hat{k}$

   [DELHI 2016]

6. Find λ, if the vectors $\vec{a} = \hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} - \hat{k}$ and $\vec{c} = \lambda\hat{j} + 3\hat{k}$ are coplanar.

   [DELHI 2015]

4 Marks Questions

7. Find the distance between the point $(-1, -5, -10)$ and the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$.

   [DELHI 2015]

8. Show that the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if $\vec{a} + \vec{b}, \vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar.

   [DELHI 2016]

9. Find the equation of the plane determined by the points $A(3, -1, 2)$, $B(5, 2, 4)$ and $C(-1, -1, 6)$ and hence find the distance between the plane and the point $P(6, 5, 9)$.

   [ALL INDIA 2012]

10. Show that the four points A, B, C and D with position vectors $4\hat{i} + 5\hat{j} + \hat{k}, -\hat{j} - \hat{k}, 3\hat{i} + 9\hat{j} + 4\hat{k}$ and $4(-\hat{i} + \hat{j} + \hat{k})$ are respectively coplanar.

    [ALL INDIA 2014]

11. Find the coordinates of the point, where the line $\frac{x-2}{3}=\frac{y-1}{4}=\frac{z-2}{2}$ intersects the plane $x-y+z-5=0$. Also find the angle between the line and the plane.

[ALL INDIA 2013]

12. Find the distance of the point $(-1,-5,-10)$ from the point of intersection of the line $\vec{r}=2\hat{i}-\hat{j}+2\hat{k}+\lambda(3\hat{i}+4\hat{j}-2\hat{k})$ and the plane $\vec{r}.\left(\hat{i}-\hat{j}+\hat{k}\right)=5$

[ALL INDIA 2018]

6 Marks Questions

13. Show that the following two lines are co-planar:

$$\frac{x-a+d}{a-\delta}=\frac{y-a}{\alpha}=\frac{z-a-d}{\alpha+\delta}$$

and $\frac{x-b+c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma}$

[ALL INDIA 2015]

14. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ then

(a) Let $c_1=1$ and $c_2=2$ find which makes $\vec{a},\vec{b}$ and $\vec{c}$ coplanar.

(b) If $c_2=-1$ and $c_3=1$, show that no value of c_1 can make $\vec{a},\vec{b}$ and $\vec{c}$ coplanar.

[DELHI 2017]

15. Find the equation of the plane through the line of intersection of the planes $x+y+z=1$ and $2x+3y+4z=5$ which is perpendicular to the plane $x-y+z=0$. Also find the distance of the plane obtained above, from the origin

[ALL INDIA 2012]

16. Find the equation of the plane which contains the line of intersection of the planes.

$\vec{r}.\left(\hat{i}-2\hat{j}+3\hat{k}\right)-4=0$ and $\vec{r}.\left(-2\hat{i}+\hat{j}+\hat{k}\right)+5=0$.

And whose intercept on x-axis is equal to that of on y-axis.

[ALL INDIA 2016]

17. Find the coordinates of the point where the line through the points (3, –4, –5) and (2, –3, 1), crosses the plane determined by the points (1, 2, 3), (4, 2, –3) and (0, 4, 3).

[ALL INDIA 2017]

18. A variable plane which remains at a constant distance $3p$ from the origin cuts the coordinate axes at A,B,C. Show that the locus of the centroid of triangle ABC is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$

[ALL INDIA 2017]

19. Find the equation of the plane through the line of intersection of $\vec{r}\left(2\hat{i}-3\hat{j}+4\hat{k}\right)=1$ and $\vec{r}\left(\hat{i}-\hat{j}\right)+4=0$ and perpendicular to the plane $\vec{r}\left(2\hat{i}-\hat{j}+\hat{k}\right)+8=0$. Hence find whether the plane thus obtained contains the line $x-1=2y-4=3z-12$.

[DELHI 2017]

20. Find the coordinate of the point P where the line through $A(3,-4,-5)$ and $B(2,-3,1)$ crosses the plane passing through three points $L(2,2,1)$, $M(3,0,1)$ and $N(4,-1,0)$. Also, find the ratio in which P divides the line segment AB.

[DELHI 2016]

21. Find the distance between the point (7, 2, 4) and the plane determined by the points

$A(2,5,-3), B(-2,-3,5)$ *and* $C(5,3,-3)$.

[DELHI 2014]

22. Find the equation of the plane which contains the line of intersection to the planes $\vec{r},\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4=0$, $\vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)+5=0$ and which is perpendicular to the plane $\vec{r},\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$.

[DELHI 2011]

23. Find the equation of the plane determined by the points $A(3,-1,2)$, $B(5,2,4,)$ and $C(-1,-1-6)$ and hence find the distance between the plane and point $P(6,5,9)$

[DELHI 2012]

24. Find the vector equation of the plane passing through the intersection of the planes $\vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=7$ and $\vec{r}.\left(2\hat{i}+5\hat{j}+3\hat{k}\right)=9$ and through the point (2, 1, 3).

[ALL INDIA 2011]

25. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.\left(\hat{i}+3\hat{j}\right)-6=0$ and $\vec{r}.\left(3\hat{i}-\hat{j}-4\hat{k}\right)=0$, whose perpendicular distance from origin is unity.

[ALL INDIA 2013]

26. Find the vector equation of the line passing through the point (1, 2, 3) and parallel to the planes $\vec{r}.\left(\hat{i}-\hat{j}-2\hat{k}\right)=5$ and $\vec{r}.\left(3\hat{i}+\hat{j}+\hat{k}\right)=6$.

[DELHI 2013]

27. Find the distance of the point $(2,\ 12,\ 5)$ from the point of intersection of the line $\vec{r}=\left(2\hat{i}-4\hat{j}+2\hat{k}\right)+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right)$ and the plane $\vec{r}.\left(\hat{i}-2\hat{j}+\hat{k}\right)=0$.

[ALL INDIA 2014]

28. Find the direction ratios of the normal to the plane, which passes through the points $(1,\ 0,\ 0)$ and $(0,\ 1,\ 0)$ and makes angle $\frac{\pi}{4}$ with the plane $x+y=3$. Also find the equation of the plane.

[ALL INDIA 2015]

29. If lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect, then find the value of k and hence find the equation of the plane containing these lines.

[DELHI 2015]

30. Find the vector equation of the plane passing through three points with vectors $\hat{i}+\hat{j}-2\hat{k},\ 2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$. Also find the coordinates of the intersection of this plane and the line $\vec{r}=3\hat{i}-\hat{j}-\hat{k}+\lambda\left(2\hat{i}-2\hat{j}+\hat{k}\right)$

[DELHI 2013]

Solutions

1. $2x-y+2z=5 \quad (1)$

$5x-2\cdot5y+5z=20$ or

$2x-y+2z=8 \quad (2)$

Distance between plane (1) & (2)

$$=\left|\frac{d_1-d_2}{\sqrt{a^2+b^2+c^2}}\right|$$

$$=\left|\frac{3}{\sqrt{9}}\right|=1$$ [1]

2. Given, Plane $\vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)-5=0$

$\Rightarrow\left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(2\hat{i}+\hat{j}-\hat{k}\right)-5=0$

$\Rightarrow 2x+y-z=5$

$\Rightarrow \frac{x}{\frac{5}{2}}+\frac{y}{5}+\frac{z}{-5}=1$

$\Rightarrow$ Sum of intercepts $=\frac{5}{2}+5-5=\frac{5}{2}$ [1]

3. Direction Ratios of normal to the plane are 5, –4,7

Direction ratios of y- axis: 0,1,0

If θ is the angle between the plane and y - axis, then

$$\sin\theta=\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{\left(a_1^2+b_1^2+c_1^2\right)}\sqrt{\left(a_2^2+b_2^2+c_2^2\right)}}\right|$$ [½]

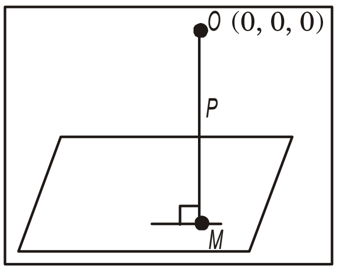
$$=\left|-\frac{4}{3\sqrt{10}}\right|$$

$$\theta=\sin^{-1}\left(\frac{4}{3\sqrt{10}}\right)$$

$\therefore$ Acute angle is $\sin^{-1}\left(\frac{4}{3\sqrt{10}}\right)$ [½]

4. The length of the perpendicular drawn from a point is given by

$$p=\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$$



[½]

$2x-3y+6z+21=0$

$$p=\left|\frac{0+0+0+21}{\sqrt{2^2+3^2+6^2}}\right|$$

$\Rightarrow p = \frac{21}{\sqrt{49}}$

$\Rightarrow p = \frac{21}{7}$

$\Rightarrow p = 3$ [½]

5. $\vec{n} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

$|\vec{n}| = \sqrt{4+9+36} = 7$

$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}$

$= \frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$

Vector equation of a plane $\vec{r}.\hat{n} = d$,

$\vec{r}\left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right) = 5$ [1]

6. Vector $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

$\begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & -1 \\ 0 & \lambda & 3 \end{vmatrix} = 0$ [1]

Expanding along C_1.

$1(-3+\lambda) - 2(9-\lambda) = 0$ [1]

$-3 + \lambda - 18 + 2\lambda = 0$

$3\lambda = 21 \Rightarrow \lambda = \frac{21}{3} = 7$

7. Equation of given line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12} = p$

Let point $B(3p+2, 4p-1, 12p+2)$ ...(i) be the point of intersection between line and plane. [1]

Point B lies on the given plane $x - y + z = 5$ [1]

$\therefore 3p + 2 - (4p-1) + (12p+2) = 5$

$\therefore 3p + 2 - 4p + 1 + 12p + 2 = 5 \Rightarrow p = 0$ [1]

Putting the value of p in (i),

Coordinates of $B(0+2, 0-1, 0+2)$ or $B(2,-1,2)$

Let $A = (-1,-5,-10)$

Required distance

$= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2}$

$= \sqrt{9+16+144}$

$= \sqrt{169} = 13\text{units}$ [1]

8. Given that $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are coplanar.

$\therefore \left[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}\right] = 0$ [1]

i.e. $(\vec{a}+\vec{b}).\left\{(\vec{b}+\vec{c}) \times (\vec{c}+\vec{a})\right\} = 0$

$(\vec{a}+\vec{b}).\left\{(\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{a})\right\} = 0$ [1]

$\Rightarrow \vec{a}.(\vec{b}\times\vec{c}) + \vec{a}.(\vec{b}\times\vec{a}) + \vec{a}.(\vec{c}\times\vec{a}) + \vec{b}.(\vec{b}\times\vec{c}) +$

$\vec{b}.(\vec{b}\times\vec{c}) + \vec{b}.(\vec{b}\times\vec{a}) + \vec{b}.(\vec{c}\times\vec{a}) = 0$ [1]

$\Rightarrow 2\left[\vec{a}, \vec{b}, \vec{c}\right] = 0$ or $\left[\vec{a}, \vec{b}, \vec{c}\right] = 0$

$\Rightarrow \vec{a}, \vec{b}, \vec{c}$ are coplanar. [1]

9. The equation of the plane through three non-collinear points A(3, -1, 2), B(5, 2, 4) and C (-1, -1, 6) can be expressed as

$\begin{vmatrix} x-3 & y+1 & z-2 \\ 5-3 & 2+1 & 4-2 \\ -1-3 & -1+1 & 6-2 \end{vmatrix} = 0$ [1]

$\Rightarrow \begin{vmatrix} x-3 & y+1 & z-2 \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 0$ [1]

$\Rightarrow 12(x-3) - 16(y+1) + 12(z-2) = 0$

$\Rightarrow 12x - 16y + 12z - 76 = 0$ [1]

$\Rightarrow 3x - 4y + 3z - 19 = 0$ is the required equation.

Now, distance of P (6, 5, 9) from the plane is given by

$= \left|\frac{3\text{x}6 - 4(5) + 3(9) - 19}{\sqrt{9+16+9}}\right| = \left|\frac{6}{\sqrt{34}}\right| = \frac{6}{\sqrt{34}}$ units [1]

10. If position vector of $\vec{A} = 4\hat{i} + 5\hat{j} + \hat{k}$,

$\vec{B} = -\hat{j} - \hat{k}$

$\vec{C} = 3\hat{i} + 9\hat{j} + 4\hat{k}$

$\vec{D} = 4(-\hat{i} + \hat{j} + \hat{k})$ [1]

Points, $\vec{A}, \vec{B}, \vec{C}, \vec{D}$ all are coplanar if

$\left[\overrightarrow{AB}\overrightarrow{AC}\overrightarrow{AD}\right] = 0$ .....(1)

So, $\overrightarrow{AB}$ = position vector of $\vec{B}$– position vector of $\vec{A} = -4\hat{i} - 6\hat{j} - 2\hat{k}$ [1]

$\overrightarrow{AC}$ = position vector of $\vec{C}$ – position vector of $\vec{A} = -\hat{i} + 4\hat{j} + 3\hat{k}$

$\overrightarrow{AD}$ = position vector of $\vec{D}$ – position vector of $\vec{A} = -8\hat{i} - \hat{j} + 3\hat{k}$

So, for $\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$ [1]

Expand along R_1 we get

$= -4[12+3] + 6[-3+24] - 2[1+32]$

$= -60 + 126 - 66 = 0$

So, we can say that point A, B, C, D are Coplanar ....Hence proved. [1]

11. Given: part I: $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} = p$

Let point $A(3p+2, 4p-1, 2p+2)$ [1]

be the required point of intersection between

The given line and plane

Then point A line in the given plane

$x - y + z - 5 = 0$

$3p + 2 - (4p - 1) + 2p + 2 - 5 = 0$

$3p + 2 - 4p + 1 + 2p - 3 = 0$ [1]

Putting the value of p in (I) we, get required of the points ,

$A = (0+2, 0-1, 0+2) = (2, -1, 2)$

Part (ii) D.rs of the given line are 3,4,2 D.rs of the normal given plane are 1,-1,1

Let θ be the required angle.

$$\sin\theta = \left|\frac{a_1 a_2 + b_1 b + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$ [1]

$$= \left|\frac{3(1) + 4(-1) + 2(1)}{\sqrt{3^2 + 4^2 + 2^2} \times \sqrt{(1)^2 + (-1)^2 + (1)^2}}\right|$$

$$\sin\theta = \left|\frac{3 - 4 + 2}{\sqrt{9 + 16 + 4} \times \sqrt{3}}\right|$$

$$\sin\theta = \frac{1}{\sqrt{87}}$$

$\theta = \sin^{-1}\left(\frac{1}{\sqrt{87}}\right) or \sin^{-1}\left(\frac{\sqrt{87}}{87}\right).$ [1]

12. The equation of the given line is

$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ ...(1)

The equation of the given plane is

$\vec{r}.\left(\hat{i} - \hat{j} + \hat{k}\right) = 5$ ...(2) [1]

Substituting the value of r from equation (1) in equation (2), we obtain

$\left[2\hat{i} - \hat{j} + 2\hat{k} + \lambda\left(3\hat{i} + 4\hat{j} + 2\hat{k}\right)\right].\left(\hat{i} - \hat{j} + \hat{k}\right) = 5$

$\Rightarrow \left[(3\lambda + 2)\hat{i} + (4\lambda - 1)\hat{j} + (2\lambda + 2)\hat{k}\right].\left(\hat{i} - \hat{j} + \hat{k}\right) = 5$

$\Rightarrow (3\lambda + 2) - (4\lambda - 1) + (2\lambda + 2) = 5$

$\Rightarrow \lambda + 5 = 5$ [1]

Substituting this value in equation (1), we obtain the equation of the line as

$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k}$

This means that the position vector of the point of intersection of the line and the plane is

$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k}$ [1]

This shows that the point of intersection of the given line and plane is given by the coordinates, (2, –1, 2). The point is (–1, –5, –10).

The distance d between the points, (2, –1, 2) and (–1, –5, –10), is

$d = \sqrt{(-1-2)^2 + (-5+1)^2 + (-10-2)^2}$

$= \sqrt{9 + 16 + 144}$

$= \sqrt{169}$

= 13 units [1]

13. $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$ [1]

Here $= \begin{vmatrix} b-c-(a-d) & b-a & b+c-(a+d) \\ a-\delta & \alpha & a+\delta \\ \beta & \beta & \beta+\gamma \end{vmatrix}$ [2]

Using $C_1 \to C_1 + C_3$

$= 2\begin{vmatrix} b-a & b-a & b+c-a-d \\ a & \alpha & a+\delta \\ \beta & \beta & \beta+\gamma \end{vmatrix} = 0$

$\therefore$ As C_1 and C_2 are identical

= 0

Hence given lines are coplanar. [1]

14. $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar

$\therefore \left[\vec{a}\,\vec{b}\,\vec{c}\right] = 0$ [1]

Then, $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$ [1]

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & c_3 \end{vmatrix} = 0 \qquad [\because c_1 = 1, c_2 = 2]$

Expanding along R_1, we have

$\Rightarrow 1(0) - 1(c_3) + 1(2) = 0$

$\Rightarrow -c_3 = -2 \quad \Rightarrow c_3 = 2$ [1]

(b) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$ [1]

$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & -1 & 1 \end{vmatrix} = 0 \qquad [\because c_2 = -1, c_3 = 1]$ [1]

Expanding along R_1, we have

$\Rightarrow 1(0) - 1(1) + 1(-1) = 0$

$\Rightarrow -1 - 1 = 0 \quad \Rightarrow -2 = 0,$ Not possible

Hence, no value of c_1 can make $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar. [1]

15. Eq^n of given planes are $P_1 : x + y + z - 1 = 0$

$P_2 : x + 3y + 4z - 5 = 0$

Equation of plane through the line of intersection of planes P_1, P_2 is $P_1 + \lambda P_2 = 0$ [1]

$(x + y + z - 1) + \lambda(2x + 3y + 4z - 5) = 0$

$(1 + 2\lambda)x + (1 + 3\lambda)y + (1 + 4\lambda)z + (-1 - 5\lambda) = 0$ ...(1)

[1]

Given that plane represented by eq^n (1) is perpendicular to plane

$x - y + z = 0$ [1]

So we use formula $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

So $(1 + 2\lambda)(1) + (1 + 3\lambda)(-1) + (1 + 4\lambda)(1) = 0$

$1 + 2\lambda - 1 - 3\lambda + 1 + 4\lambda = 0$

$3\lambda + 1 = 0$ [1]

$\lambda = -\dfrac{1}{3}$

Put in eq^n (1) so we get

$\left(1 - \dfrac{2}{3}\right)x + (1 - 1)y + \left(1 - \dfrac{4}{3}\right)z + \dfrac{2}{3} = 0$ [1]

$\dfrac{x}{3} - \dfrac{z}{3} + \dfrac{2}{3} = 0$

$x - z + 2 = 0$ [1]

16. Planes are $\vec{r}.\left(\hat{i} - 2\hat{j} + 3\hat{k}\right) - 4 = 0$

$\vec{r}.\left(-2\hat{i} + \hat{j} + \hat{k}\right) + 5 = 0$

$\Rightarrow x - 2y + 3z - 4 = 0$ [1]

And $-2x + y + z + 5 = 0$

Any plane passing through the line of intersect

$(x - 2y + 3z - 4) + \lambda(-2x + y + z + 5) = 0$ [1]

$x(1 - 2\lambda) + y(-2 + \lambda) + z(3 + \lambda) + (5\lambda - 4) = 0$

Intercepts are equal on axes

So $\dfrac{4 - 5\lambda}{1 - 2\lambda} = \dfrac{4 - 5\lambda}{-2 + \lambda}$ [2]

$\Rightarrow -2 + \lambda = 1 - 2\lambda$

$\Rightarrow 3\lambda = 3$

$\Rightarrow \lambda = 1$ [1]

Required plane

$-x-y+4z+1=0$

$\Rightarrow x+y-4z-1=0$ [1]

17. Equation of line passing through

$(3,\ -4,\ -5)$ and $(2,\ -3,\ 1)$

$\frac{z-3}{1}=\frac{y+4}{1}=\frac{z+5}{6}$ ...(1) [1]

Equation of plane passing through

$(1,\ 2,\ 3),\ (4,\ 2,\ -3)$ and $(0,\ 4,\ 3)$

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 3 & 0 & -6 \\ -1 & 2 & 0 \end{vmatrix}=0$$ [1]

$\Rightarrow (x-1)(12)-(y-2)(-6)+(z-3)(6)=0$

$\Rightarrow 2x + y + z - 7 = 0$ ...(2) [1]

Let any point on line (1)

is $P(-k+3,\ k-4,\ 6k-5)$ [1]

it lies on plane

Thus, $2(-k+3)+k-4+6k-5-7=0$

$5k = 10$

$\Rightarrow k = 2$ [1]

Thus, $P(1,\ -2,\ 7)$ [1]

18. Let the equation of plane

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ (1) [1]

It cut the co-ordinate axes at A, B and C

Thus, $A(a,\ 0,\ 0), B\ (0,\ b,\ 0), C(0,\ 0,\ c)$ [1]

Let the centroid of ΔABC be (x, y, z)

given that distance of plane (1) from origin is $3p$ [1]

$$\therefore \left|\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right|=3p$$ [1]

$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{9p^2}$ [1]

From (2)

$\Rightarrow \frac{1}{9x^2}+\frac{1}{9y^2}+\frac{1}{9z^2}=\frac{1}{9p^2}$

$\Rightarrow \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$ [1]

19. Given planes

$\vec{r}(2\hat{i}-3\hat{j}+4\hat{k})=1$

$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(2\hat{i}-3\hat{j}+4\hat{k})=1$

$2x-3y+4z-1=0$ ... (1) [1]

Similarly, $\vec{r}(\hat{i}-\hat{j})+4=0$

$x-y+4=0$ ... (2)

Required equation of plane is

$(2x-3y+4z-1)+p(x-y+4)=0$ ... (3)

$(2+p)x+(-3-p)y+4z-1+4p=0$ ... (4) [1]

D.rs of normal to the plane are

$2+p, -3-p, 4$

D.rs of normal to the plane

$\vec{r}(2\hat{i}-\hat{j}+\hat{k})+8=0$ are $2, -1, 1$

They are perpendiculars

Using $a_1a_2+b_1b_2+c_1c_2=0$

$2(2+p)-1(-3-p)+1(4)=0$

$4+2p+3+p+4=0$

$3p=-11 \Rightarrow p=-\frac{11}{3}$ [1]

Putting the value of p in , we have

$(2x-3y+4z-1)-\frac{11}{3}(x-y+4)=0$

$\frac{6x-9y+12z-3-11x+11y-44}{3}=0$

$\Rightarrow -5x+2y+12z-47=0$ (5)

Given line:

$x-1=2y-4=3z-12$

$x-1=2(y-2)=3(z-4)$

Dividing throughout by 6

$\Rightarrow \frac{x-1}{6} = \frac{y-2}{3} = \frac{z-4}{2}$ [1]

Point $A(1,2,4)$ lies on the given line

D.rs of given line are $6,3,2$

To check whether point A lies in the plane (5) or not

$-5(1)+2(2)+12(4)-47=0$

$-5+4+48-47=0$

0 = 0, which is true.

Point A lies in the plane (5) and (6) [1]

To check whether line is perpendicular to the plane or not

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

$= -30+6+24$

= 0

Yes, given line is perpendicular to the plane (5). (7)

From (6) and (7), plane (5) contains the given line. [1]

20. Direction ratios of AB are;

2-3 , -3+4, 1+5

Or, -1, 1, 6

Equation of line using $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ is:

$\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda\,(let)$ [1]

Let $P(-\lambda+3, \lambda-4, 6\lambda-5)$ ....(i)

be the required point of intersection between line AB and plane $L(2,2,1)$, $M(3,0,1)$, $N(4,-1,0)$.

Form equation of plane using

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$ [1]

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 3-2 & 0-2 & 1-1 \\ 4-2 & -1-2 & 0-1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 1 & -2 & 0 \\ 2 & -3 & -1 \end{vmatrix} = 0$$ [1]

Expanding along R_1, we get:

$[(x-2)(2-0)-(y-2)(-1-0)+(z-1)(-3+4)] = 0$

$2x-4+y-2+z-1=0$

$2x+y+z=7$ ....(ii)

Point $P(-\lambda+3, \lambda-4, 6\lambda-5)$ lies on plane (ii)

$2(-\lambda+3) + 1(\lambda-4) + (6\lambda-5) = 7$

$-2\lambda+6+\lambda-4+6\lambda-5=7$

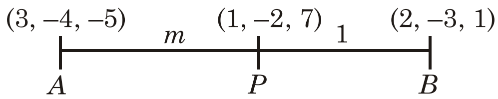
$5\lambda-3=7$

$\Rightarrow 5\lambda = 10$

$\Rightarrow \lambda = 2$ [1]

From (i), the required point of intersection is $P(-2+3, 2-4, 12-5)$

$= P(1, -2, 7)$



[1]

Let $AP : PB = m:1$

Coordinates of P = coordinates of P

$\left(\frac{2m+3}{m+1}, \frac{-3m-4}{m+1}, \frac{m-5}{m+1}\right) = (1,-2,7)$

$or, \frac{2m+3}{m+1} = 1 \Rightarrow 2m+3 = m+1$

Therefore, $m = -2$

Hence, the required ratio is m : 1 = –2 : 1 or 2 : 1 (externally). [1]

21. Let a, b, c be the direction ratios of the normal to the required palne.

Equation of plane through point $A(2, 5, -3)$ is

$a(x-2)+b(y-5)+c(z+3)=0$ ...(i)

Point B lies on (i), $-4a-8b+8c=0$ ...(ii)

Point C lies on (i), $3a-2b+0c=0$ ...(iii) [1]

Solving (ii) and (iii)

$\frac{a}{0+16} = \frac{-b}{0-24} = \frac{c}{8+24}$ [1]

Let, $\frac{a}{16} = \frac{b}{24} = \frac{c}{32} = \lambda$

$\therefore a = 16\lambda, b = 24\lambda, c = 32\lambda$

Or $a = 2, b = 3, c = 4$ [1]

Putting the values of a, b and c in (i),

$2(x-2)+3(y-5)+4(z+3)=0$

$2x-4+3y-15+4z+12=0$

$2x+3y+4z-7=0$ [1]

Required distance $=\left|\dfrac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$ [1]

$=\left|\dfrac{2(7)+3(2)+4(4)-7}{\sqrt{2^2+3^2+4^2}}\right|$

$=\left|\dfrac{14+6+16-7}{\sqrt{4+9+16}}\right|$

$=\dfrac{29}{\sqrt{29}}=\sqrt{29}$ *units* [1]

22. Equations of given planes are

$\vec{r}.(\hat{i}+2\hat{j}+3\hat{k})=4,\ \vec{r}.(2\hat{i}+\hat{j}-\hat{k})=-5$

Here $\vec{n}_1=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{n}_2=2\hat{i}+\hat{j}-\hat{k}$ [1]

$d_1=4$ $\quad d_2=-5$

Using $\vec{r}.(\vec{n}_1+\lambda\vec{n}_2)=d_1+\lambda d_2$

$\vec{r}.\left[(\hat{i}+2\hat{j}+3\hat{k})+\lambda(2\hat{i}+\hat{j}-\hat{k})\right]$

$=4+\lambda(-5)$ [1]

$\vec{r}.\left[(1+2\lambda)\hat{i}+(2+\lambda)\hat{j}+(3-\lambda)\hat{k}\right]=4-5\lambda$

$(x\hat{i}+y\hat{j}+z\hat{k}).\left[(1+2\lambda)\hat{i}+(2+\lambda)\hat{j}+(3-\lambda)\hat{k}\right]$

$=4-5\lambda$

[Using $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$]

$(1+2\lambda)x+(2+\lambda)y+(3-\lambda)z=4-5\lambda$ [1]

Is the equation of plane through the intersection of the given two planes.

D.rs of the normal to the plane (ii) are $1+2\lambda, 2+\lambda, 3-\lambda$

D.rs of the normal to the given 3rd plane are

$\vec{r}.(5\hat{i}+3\hat{j}-6\hat{k})+8=0$ *are* $5, 3, -6$ [1]

Using $a_1a_2+b_1b_2+c_1c_1=0$

$5(1+2\lambda)+3(2+\lambda)-6(3-\lambda)=0$

$5+10\lambda+6+3\lambda-18+6\lambda=0$

$19\lambda=7 \quad \Rightarrow \lambda=\dfrac{7}{19}$ [1]

Putting the value of λ in Eq.(i),

$\vec{r}\left[\left(1+2\left(\dfrac{7}{19}\right)\right)\hat{i}+\left(2+\dfrac{7}{19}\right)\hat{j}+\left(3-\dfrac{7}{19}\right)\hat{k}\right]$

$=4-5\left(\dfrac{7}{19}\right)$

$\vec{r}.\left[\dfrac{33}{19}\hat{i}+\dfrac{45}{19}\hat{j}+\dfrac{50}{19}\hat{k}\right]=\dfrac{41}{19}$

$\vec{r}.(33\hat{i}+45\hat{j}+50\hat{k})=41$ [1]

23. Given: Equation of plane passing through three points: $A(3,-1,2), B(5,2,4), C(-1,-1,6)$ *is*

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$ [1]

$$\Rightarrow \begin{vmatrix} x-3 & y+1 & z-2 \\ 5-3 & 2+1 & 4-2 \\ -1-3 & -1+1 & 6-2 \end{vmatrix}=0$$ [1]

$$\Rightarrow \begin{vmatrix} x-3 & y+3 & z-2 \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix}=0$$ [1]

Expanding along R_1.

$\Rightarrow (x-3)(12-0)-(y+1)(8+8)+(z-2)(0+12)=0$

$\Rightarrow 12(x-3)-16(y+1)+12(z-2)=0$

$\Rightarrow 3(x-3)-4(y+1)+3(z-2)=0$

$\Rightarrow 3x-9-4y-4+3z-6=0$

$\Rightarrow 3x-4y+3z-19=0$ [1]

The $\perp$ distance of the point $P(6, 5, 9)$ plane (i) is

$\left|\dfrac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$

$a = 3, b = 4, c = 3, d = -19$

$= \left|\frac{3(6) - 4(5) + 3(9) - 19}{\sqrt{9+16+9}}\right|$ [1]

$= \left|\frac{18 - 20 + 27 - 19}{\sqrt{34}}\right|$

$= \left|\frac{6}{\sqrt{34}}\right|$

$= \frac{6}{\sqrt{34}} \times \frac{\sqrt{34}}{\sqrt{34}}$

$= \frac{6\sqrt{34}}{34} = \frac{3\sqrt{34}}{17}$ units. [1]

24. Equations of planes are:

$\vec{r}.(2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$ and $\vec{r}.(2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$

$\vec{r}.(2\hat{i} + 2\hat{j} - 3\hat{k}) - 7 = 0$ and

$\vec{r}.(2\hat{i} + 5\hat{j} + 3\hat{k}) - 9 = 0$ [1]

The equation of plane passing through the intersection of the given plane is:

$\left[\vec{r}.(2\hat{i} + 2\hat{j} - 3\hat{k}) - 7\right] + \lambda\left[\vec{r}.(2\hat{i} + 5\hat{j} + 3\hat{k}) - 9\right] = 0$

$\vec{r}\left[(2\hat{i} + 2\hat{j} - 3\hat{k}) - 7 + \lambda(2\hat{i} + 5\hat{j} + 3\hat{k}) - 9\right] = 0$

$\vec{r}\left[(2 + 2\lambda)\hat{i} + (2 + 5\lambda)\hat{j} + (3\lambda - 3)\hat{k}\right] = 9\lambda + 7$ [1]

Point (2, 1, 3)

Position vector of plane passing through the given point is

$\vec{r} = 2\hat{i} + \hat{j} + 3\hat{k}$

Now, the equation of desired plane is:

$(\hat{2} + \hat{j} + 3\hat{k})\left[(2 + 2\lambda)\hat{i} + (2 + 5\lambda)\hat{j} + (3\lambda - 3)\hat{k}\right]$ [1]

$= 9\lambda + 7$

$(2 + 2\lambda)2 + (2 + 5\lambda) + (3\lambda - 3)3 = 9\lambda + 7$

$4 + 4\lambda + 2 + 5\lambda + 9\lambda - 9 = 9\lambda + 7$

$9\lambda = 10$

$\lambda = \frac{10}{9}$ [1]

Now,

$\vec{r}\left[(2 + 2\lambda)\hat{i} + (2 + 5\lambda)\hat{j} + (3\lambda - 3)\hat{k}\right] = 9\lambda + 7$

$\vec{r}\left[\left(2 + 2 \times \frac{10}{9}\right)\hat{i} + \left(2 + 5 \times \frac{10}{9}\right)\hat{j} + \left(2 \times \frac{10}{9} - 3\right)\hat{k}\right]$

$= 9\lambda + 7$ [1]

$\vec{r}\left[\frac{38}{9}\hat{i} + \frac{68}{9}\hat{j} + \frac{3}{9}\hat{k}\right] = 10 + 7$

$\vec{r}(38\hat{i} + 68\hat{j} + 3\hat{k}) = 153$

This is the vector equation of the required plane. [1]

25. Let $P_1: \vec{r}.(\hat{i} + 3\hat{j}) - 6 = 0$

P_1: is $x + 3y - 6 = 0$

P_2: is $\vec{r}.(3\hat{i} - \hat{j} - 4\hat{k}) = 0$

P_2: is $3x - y - 4z = 0$ [1]

Equation of plane passing through intersection of P_1 and P_2 is $P_1 + \lambda P_2 = 0$

$(x + 3y - 6) + \lambda(3x - y - 4z) = 0$

$(1 + 3\lambda)x + (3 - \lambda)y + (-4\lambda)z + (-6) = 0$ [1]

Its distance from (0, 0, 0) is 1.

$\left|\frac{0 + 0 + 0 - 6}{\sqrt{(1 + 3\lambda)^2 + (3 - \lambda)^2 + (-4\lambda)^2}}\right| = 1$ [1]

$36 = (1 + 3\lambda)^2 + (3 - \lambda)^2 + (-4\lambda)^2$

$36 = 1 + 9\lambda^2 + 6\lambda + 9 + \lambda^2 - 6\lambda + 16\lambda^2$

$36 = 26\lambda^2 + 10$

$\Rightarrow 26\lambda^2 = 26$

$\Rightarrow \lambda^2 = 1$

$\lambda = \pm 1$ [1]

Hence required plane is

For $\lambda = 1$, $(x + 3y - 6) + 1(3x - y - 4z) = 0$,

$4x + 2y - 4z - 6 = 0$ [1]

For $\lambda = -1$, $(x + 3y - 6) - 1(3x - y - 4z) = 0$,

$-2x + 4y + 4z - 6 = 0$ [1]

26. $P_1 : \vec{r}.\left(\hat{i}-\hat{j}-2\hat{k}\right)=5$

The line parallel to plane P_1 and P_2 will be perpendicular to $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ [1]

$\therefore \vec{b} \parallel \left(\overrightarrow{n_1}\times\overrightarrow{n_2}\right)$

$\overrightarrow{n_1}=\hat{i}-\hat{j}+2\hat{k}$

$\overrightarrow{n_2}=3\hat{i}+\hat{j}+\hat{k}$ [1]

$\overrightarrow{n_1}\times\overrightarrow{n_2}=-3\hat{i}+5\hat{j}+4\hat{k}$ [1]

$\therefore \vec{b}=-3\hat{i}+5\hat{j}+4\hat{k}$ [1]

Point is(1, 2, 3)

$\therefore \vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ [1]

Thus, required line is $\vec{r}=\vec{a}+\lambda\vec{b}$

$\vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(-3\hat{i}+5\hat{j}+4\hat{k}\right)$ [1]

27. The equation of the plane :

$\vec{r}.\left(\hat{i}-2\hat{j}+\hat{k}\right)=0$ [1]

The point of intersection of the line and the plane: Substituting general point of the line in the equation of plane and finding the particular value of λ. [1]

$\left[(2+3\lambda)\hat{i}+(-4+4\lambda)\hat{j}+(2+2\lambda)\hat{k}\right].(\hat{i}-2\hat{j}+\hat{k})=0$

$(2+3\lambda).1+(-4+4\lambda)(-2)+(2+2\lambda).1=0$ [1]

$12-3\lambda=0$ or $\lambda=4$

$\therefore$ The point of intersection is (14, 12, 10)

Distance of this point from $(2,\ 12,\ 5)$ is

$=\sqrt{(14-2)^2+(12-12)^2+(10-5)^2}$ [1]

{applying distance formula}

$=\sqrt{12^2+5^2}$

$=13$ [1]

28. Equation of plane passing through

$a(x-1)+b(y-0)+c(z-0)=0$

Or $ax+by+cz-a=0$ ...(1) [1]

Plane (1) passes through (0, 1, 0)

$b-a=0$ ...(2) [1]

Angle between plane (i) and plane $x+y=3$ is $\frac{\pi}{4}$

$$\therefore \cos\frac{\pi}{4}=\frac{a+b}{\sqrt{a^2+b^2+c^2}\sqrt{2}}$$

$$\Rightarrow \frac{1}{\sqrt{2}}=\frac{a+b}{\sqrt{a^2+b^2+c^2}\sqrt{2}}$$ [1]

$\Rightarrow a+b=\sqrt{a^2+b^2+c^2}$

Using (2)

$\Rightarrow 2a=\sqrt{2a^2+c^2}$

$c=\pm\sqrt{2}\,a$ [1]

$\therefore$ Equation (i) becomes

$a(x-1)+a(y-0)\pm\sqrt{2}\,a(z-0)=0$

$\Rightarrow x+y\pm\sqrt{2}z-1=0$

Direction ratio of the normal is $1,1,\pm\sqrt{2}$ [1]

29. Here,

$x_1=1, y_1=-1, z_1=1, a_1=2, b_1=3, c_1=4$

$x_2=3, y_2=k, z_2=0, a_2=1, b_2=2, c_2=1$

Given two lines intersect (given) [1]

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$ [1]

$$\begin{vmatrix} 3-1 & k+1 & 0-1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}=0$$

$$\begin{vmatrix} 2 & k+1 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}=0$$ [1]

Expanding along R_1, we have

$2(3-8)-(k+1)(2-4)-1(4-3)=0$

$-10+2k+2-1=0\Rightarrow 2k=9\Rightarrow k=\frac{9}{2}$ [1]

Equation of plane containing given lines is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

$$\begin{vmatrix} x-1 & y+1 & z-1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$ [1]

Expanding along R_1, we have

$$(x-1)(3-8)-(y+1)(2-4)+(z-1)(4-3)=0$$

$$-5x+5+2y+2+z-1=0$$

$$-5x+2y+z=-6$$

or $5x-2y-z=6$ [1]

30. Given: part I:let $\vec{a}=\hat{i}+\hat{j}-2\hat{k}$

$$\vec{b}=2\hat{i}-\hat{j}+\hat{k}$$

$$\vec{c}=\hat{i}+2\hat{j}+\hat{k}$$

Now, $\vec{b}-\vec{a}=\hat{i}-2\hat{j}+3\hat{k}, \vec{c}-\vec{a}=\hat{j}+3\hat{k}$

$$(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 0 & 1 & 3 \end{vmatrix}$$ [1]

$$=\hat{i}(-6-3)-\hat{j}(3-0)+\hat{k}(1)$$

$$=-9\hat{i}-3\hat{j}+\hat{k}$$

Vector equation of plane through

$\vec{a}$, $\vec{b}$ and $\vec{c}$ is [1]

$$(\vec{r}-\vec{a}).\left[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\right]=0$$

$$\left(\vec{r}-\left(\hat{i}+\hat{j}-2\hat{k}\right)\right).\left(-9\hat{i}-3\hat{j}+\hat{k}\right)=0$$

$$\vec{r}.\left(-9\hat{i}-3\hat{j}+\hat{k}\right)-\left(\hat{i}+\hat{j}-2\hat{k}\right).\left(-9\hat{i}-3\hat{j}+\hat{k}\right)=0$$ [1]

$$\vec{r}.\left(-9\hat{i}-3\hat{j}+\hat{k}\right)-(-9-3-2)=0$$

$$\vec{r}.\left(-9\hat{i}-3\hat{j}+\hat{k}\right)+14=0$$

Part II

Given line is

$$\vec{r}.=3\hat{i}-\hat{j}-\hat{k}+\lambda\left(2\hat{i}-2\hat{j}+\hat{k}\right)$$

equation of plane is.

$$\left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(-9\hat{i}-3\hat{j}-\hat{k}\right)+14=0$$

$$-9x-3y+z+14=0$$ [1]

Pt.(3, 1, –1) lies on the given line Direction ratios of given line are (2, –2, 1)

Cartesian equation on the given line is

$$\frac{x-3}{2}=\frac{y+1}{-2}=\frac{z+1}{1}=p$$ [1]

Let point $A(2p+3,-2p-1,p-1)$

To be the required point of intersection .

Then point A must lie in plane (i)

$$-9(2p+3)-3(-2p-1)+p-1+14=0$$

$$-18p-27+6p+3+p-1+14=0$$

$$-11p-11=0$$

$$\Rightarrow p=-1$$

Putting the value of p in (ii),

Point $A(-2+3,2-1,-1-1)=(1,1,-2)$

The required point is (1, 1, –2) [1]

Smart Notes ....

CHAPTER 12

Linear Programming

Chapter Analysis of Previous Three Years' Board Exams

Number of Questions Asked in Exams

Topics	2018		2017		2016	
	Delhi	**All India**	**Delhi**	**All India**	**Delhi**	**All India**
LPP	6 marks	6 marks	2, 4 marks	2, 4 marks	6 marks	6 marks

Summary

- **Linear Programming**

  Linear programming is a method which provides the optimization (maximization or minimization) of a linear function composed of certain variables subject to the number of constraints.

- **Applications of Linear Programming**
  - Used in finding highest margin, maximum profit, minimum cost etc.
  - Used in industry, commerce, management science etc.

- **Linear Programming Problem (LPP)**

  Linear Programming problem is a type of problem in which a linear function z is maximized or minimized on certain conditions that are determined by a set of linear inequalities with non-negative variables.

- **Mathematical Formulation of LPP**
  - **Optimal value:** Maximum or Minimum value of a linear function
  - **Objective Function:** The function which is to be optimized (maximized/minimized).
  - **Linear objective function:** $Z = ax + by$ is a linear function form, where a, b are constants, which has to be maximized or minimized is called a linear objective function.

    For example- $Z = 340x + 60y$, where variables x and y are called **decision variables**.

- **Constraints:** The limitations as disparities on the factors of a LPP are called constraints. The conditions $x \geq 0$, $y \geq 0$ are called **non-negative restrictions**.

- **'Linear'** states that all mathematical relations used in the problem are linear relations. **Programming** refers to the method of determining a particular program or plan of action.

- **Mathematical Formulation of the Problem**

  A general LPP can be stated as (Max/Min) $Z = c_1x_1 + c_2x_2 + \dots\dots + c_nx_n$ subject to given constraints and the non-negative restrictions.
  - $x_1, x_2, \dots\dots, x_n \geq 0$ and all are variables.
  - $c_1, c_2, \dots\dots c_n$ are constants.

- **Graphical methods to solve a Linear Programming Problem**

- **Corner Point method:** This method is used to solve the LPP graphically by finding the corner points.

  Procedure-

  (i) Replace the signs of inequality by the equality and consider each constraint as an equation.

  (ii) Plotting each equation on the graph that will represent a straight line.

  (iii) The common region that satisfies all the constraints and the non-negative restrictions is known as the **feasible region**. It is a convex polygon.

  (iv) Determining the vertices of the convex polygon. These vertices of the polygon are also known as the extreme points or corners of the feasible region.

  (v) Finding the values of Objective function at each of the extreme points. Now, finding the point at which the value of the objective function is optimum as that is the optimal solution of the given LPP.

- **General features of a LPP**
  - The feasible region is always a convex region.
  - The maximum (or minimum) solution of the objective function occurs at the vertex (corner) of the feasible region.
  - If two corner points produce the same optimum (maximum or minimum) value of the objective function, then every point on the line segment joining these points will also give the same optimum (maximum or minimum) value.

- **Different Types of Linear Programming Problems**
  - **Manufacturing problems**

    In order to make maximum profit, determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed manpower, machine hours, warehouse space per unit of the output etc., in order to make maximum profit.
  - **Diet problems**

    Determining the minimum amount of different nutrients which should be included in a diet so as to minimize the cost of the diet.
  - **Transportation problems**

    To find the cheapest way of transporting a product from factories situated at different locations to different markets.
  - **Allocation problems**

    These problems are concerned with the allocation of a particular land/area of a company or any organization by choosing a certain number of employees and a certain amount of area to complete the assignment within the required deadline, given that a single person works on only one job within the assignment.
- Integration is the inverse of differentiation. Instead of diff

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

4 Marks Questions

1. A small firm manufactures necklace and bracelets. The total number of necklaces and bracelets that it can handle per day is at most 24. It takes 1 hour to make a bracelet and half an hour make a necklace. The maximum number of hours available per day is 16. If the profit on a necklace is Rs100 and that on a bracelet is Rs300, formulate an L.P.P for finding how many of each should be produced daily to maximize the profit? It is being given that at least one of each must be produced.

   [DELHI 2017]

2. Solve the following L.P.P. graphically:

   Minimise $Z = 5x + 10y$

   Subject to $x + 2y \leq 120$

   Constraints: $x + y \geq 60$, $x - 2y \geq 0$ and x,y ≥ 0

   [DELHI 2017]

3. A cooperative society of farmers has 50 hectares of land to grow two crops A and B The profit from crops A and B per hectare are estimated as Rs. 10,500 B and Rs. 9,000 respectively. To control weeds a liquid herbicide has to be used for crops A and B at the rate of 20 litres and 10 litres per hectare, respectively. Further not more that 800 litres of herbicide should be used in order to protect fish and wildlife is more important drainage from this land. Keeping in mind that the protection of fish and other wildlife is more important than earing profit ,how much land should be allocated to each crop so as to maximize the total profit ? From an LPP from the above and solve it graphically.

   [ALL INDIA 2013]

6 Marks Questions

4. A manufacturer produces two products A and B. Both the products are processed on two different machines. The available capacity of first machine is 12 hours and that of second machine is 9 hours per day. Each unit of product A requires 3 hours on both machines and each unit of product B requires 2 hours on first machine and 1 hour on second machine, Each unit of product A is sold at Rs 7 profit and that of B at a profit of Rs 4. Find the production level per day for maximum profit graphically.

   [DELHI 2016]

5. A factory makes tennis rackets and cricket bats. A tennis racket 1.5 hours of machine time and 3 hours of craftsman 's time in its making while a cricket bat takes 3 hours of machine time and 1 hours of craftsman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftsman's time. If the profit on a racket and on a bat is Rs. 20 and Rs. 10 respectively, find the number of tennis rackets and cricket bats that the factory must manufacture to earn the maximum profit . Make it as an L.P.P and solve it graphically.

[DELHI 2011]

6. A manufacturer produces nuts and bolts . It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hours on machine B to produce a package of bolts . He earns a profit of Rs. 17.50 per package on nuts and Rs. 7 per package on bolts . How many package of each should be produced each day so as to maximize his profits if he operates his machine for at the most 12 hours a day? Form the above as a linear programming problem and solve it graphically.

[DELHI 2012]

7. Dealer in a rural area wishes to purchase a number of sewing machines. He has only Rs 5,760 to invest and has space for at most 20 items for storage. An electronic sewing machine cost him Rs 360 and a manually operated sewing machine Rs 240. He can sell an electronic sewing machine at a profit of Rs 22 and a manual operated sewing machine at a profit of Rs 18. As you make that he can sell old items that he can buy, how should he invest his money in order to maximize his profit? Make it as a LPP and solve it graphically.

[DELHI 2014]

8. A factory manufactures two types of screws, A and B. Each type of screw requires the use of two machines, an automatic and a hand operated. It takes 4 minutes on the automatic and 6 minutes on hand operated machines to manufacture a package of screws A, while it takes 6 minutes on automatic and 3 minutes on the hand operated machines to manufacture a package of screws B. Each machine is available for at the most 4 hours on any day. The manufacturer can sell a package of screws A at a profit of 70 paise and screws B at a profit of Rs1. Assuming that he can sell all the screws he manufactures, how many packages of each type should the factory owner produce in a day in order to maximize his profit? Formulate the above LPP and solve it graphically and find the maximum profit.

[DELHI 2018]

9. A manufacturer considers that men and women workers are equally efficient and so he pays them at the same rate. He has 30 and 17 units of workers (male and female) and capital respectively, which he uses to produce two types of goods A and B. To produce one unit of A, 2 workers and 3 units of capital are required while 3 workers 1 unit of capital is required to produce one unit of B. If A and B are priced at Rs.100 and Rs.120 per unit respectively, how should he use his resource to maximize the total revenue? Form the above as an LPP and solve graphically. Do you agree with this view of the manufacturer that men and women workers are equally efficient and so should be paid at same rate?

[ALL INDIA 2013]

10. A manufacturing company makes two types of teaching aids A and B of Mathematics for class XII. Each type of A requires 9 labour hours of fabricating and 1 labour hour for finishing. Each type of B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available per week are 180 and 30 respectively. The company makes a profit of Rs 80 on each piece of type A and Rs. 120 on each piece of type B. How many pieces of type A and type B should be manufactured per week to get a maximum profit? Make it as an LPP and solve graphically. What is the maximum profit per week?

[ALL INDIA 2014]

11. The postmaster of a local post office wishes to hire extra helpers during the Deepawali season, because of a large increase in the volume of mail handling and delivery. Because of the limited office space and the budgetary conditions, the number of temporary helpers must not exceed 10. According to past experience, a man can handle 300 letters and 80 packages per day, on the average, and a woman can handle 400 letters and 50 packets per day. The postmaster believes that the daily volume of extra mail and packages will be no less than 3400 and 680 respectively. A man receives Rs.225 a day and a woman receives Rs. 200 a day. How many men and women helpers should be hired to keep the pay-roll at a minimum? Formulate an LPP and solve it graphically.

[ALL INDIA 2015]

12. A retired person wants to invest an amount of Rs.50,000. His broker recommends investing in two types of bonds 'A' and 'B' yielding 10% and 9% return respectively on the invested amount. He decides to invest at least Rs.20,000 in bond 'A'and at least Rs.10,000 in bond 'B'. He also wants to invest at least as much in bond 'A' as in bond 'B'. Solve this linear programming problem graphically to maximize his returns.

[ALL INDIA 2016]

13. Two tailors, A and B, earn Rs 300 and Rs 400 per day respectively. A can stitch 6 shirts and 4 pairs of trousers while B can stitch 10 shirts and 4 pairs of trousers per day. To find how many days should each of them work and if it is desired to produce at least 60 shirts and 32 pairs of trousers at a minimum labour cost, formulate this as an LPP.

[ALL INDIA 2017]

14. Find graphically, the maximum value of $Z = 2x + 5y$, subject to constraints given below:

$2x + 4y \le 8$, $3x + y \le 6$, $x + y \le 4$,

$x \ge 0, y \ge 0$

[DELHI 2015]

Solutions

1. Let x be the number of necklaces and y be the number of bracelets.

	Time to make	Profit
X	$\frac{1}{2}$ Hour	Rs 100
Y	1 Hour	Rs 300
Total available	16	

[1]

Maximum profit $Z = Rs(100x + 300y)$.

Subject to the constraints:

$x + y \le 24$ [1]

$\frac{1}{2}x + y \le 16$ [1]

$\Rightarrow x + 2y \le 32$

$x \ge 1,\ y \ge 1$ [1]

2. Given : $x + 2y \le 120$,

$x + y \ge 60$, $x - 2y \ge 0$ and $x, y \ge 0$

Using $x + 2y \le 120$

X	0	120
Y	60	0

[1]

$x + y \ge 60$

X	0	60
y	60	0

$x - 2y \ge 0$

X	0	60	40
y	0	30	20

Changing the inequality $x + 2y = 120$ ...(1)

$x - 2y = 0$ ...(2)

Subtracting (2) from (1)

$x+2y-x+2y=120-0$

$4y=120$

$y=30$

Putting value of y in 1 we get

$x+2(30)=120$

$x+60=120$

$x=60$

Changing the inequality $x+y=60$ ...(3) [1]

$x-2y=0$ ...(4)

Subtracting (4) from (3)

$x+y-x+2y=60-0$

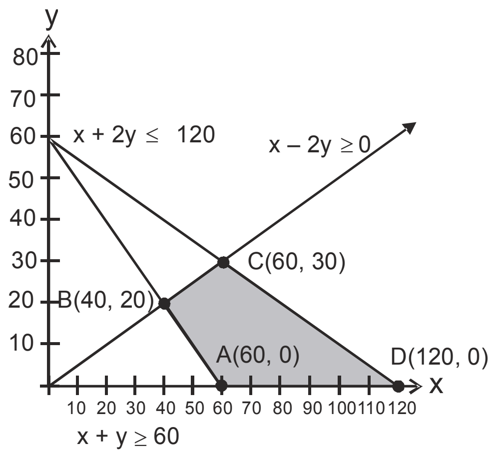
$3y=60$

$y=20$

Putting value of y in 3 we get

$x+20=60$

$x=40$



[1]

Corner points	$Z=5x+10y$
A(60,0)	**300 + 0 = 300 (Minimum)**
B(40,20)	200 + 200 = 400
C(60,30)	300 + 300 = 600
D(120,0)	600 + 0 = 600

[1]

Minimum value of Z is 300 at A(60, 0).

3. Given : let x hectare of land be allocated to crop A and y hectare of land be allocated to crop B

$Z_{max}=10500x+9000y$

Subject to the constraints

$x+y\le 50$

$20x+10y\le 800$

$\Rightarrow 2x+y\le 80$ [1]

$x\ge 0, y\ge 0$

Let $x+y=50$

x	0	50
y	50	0

[1]

Let $2x+y=80$

x	0	40
y	80	0

[1]

corner points	$Z=10,500x+9,000y$
$O(0,0)$	0
$A(40,0)$	$10,500(40)+9,000(0)=4,20,000$
$B(30,20)$	$10,500(30)+9,000(20)(=4,95,000)$
$C(0,50)$	$0+9,000(50)=4,50,000$

Here the society will get the maximum profit of Rs. 4,95,000 by allocating 30 hectares for crop A and 20 hectares for crop B. [1]

4.

	Machines			
	Quantity	**I**	**II**	**Profit**
Product A	x	3	3	Rs7
Product B	y	2	1	Rs4
		12 hours (max)	9 hours (max)	

[1]

Maximise, Z = Rs$(7x+4y)$

Subject to the constraints,

$3x+2y\le 12$

$3x+y\le 9$ $x\ge 0,\ y\ge 0$

Let $3x+2y=12$

X	0	4	2
Y	6	0	3

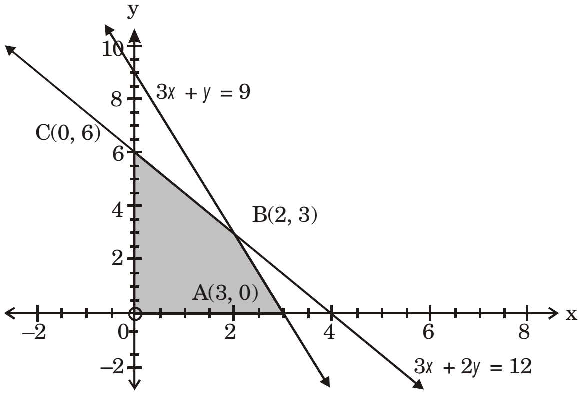
[1]

Let $3x+y=9$

X	0	3	2
Y	9	0	3

[1]

The graph using the above two equations can be drawn as:



[1]

Corner Points	$Z = Rs(7x + 4y)$
O (0,0)	0+0=0
A (3,0)	21+0=21
B (2,3)	14+12=26 **(Maximum)**
C (0,6)	0+24=24

[1]

Hence, the maximum value of Z is Rs 26 at the point (2,3).

Therefore, The production level for maximum profit is:

Product $A = 2$ units per day

Product $B = 3$ units per day

Maximum Profit = Rs 26 [1]

5. Let number of tennis rackets = x

And number of cricket bats = y

Maximise profit $Z = 20x + 10y$ [1]

Subject to the constraints,

$1.5x + 3y \leq 42$ [1]

$\Rightarrow \frac{3}{2}x + 3y \leq 42$

$\Rightarrow 3x + 6y \leq 84$

$\Rightarrow x + 2y \leq 28$

$\Rightarrow x + 2y \leq 28$

$3x + 1y \geq 24$ [1]

$x \geq 0, \ y \geq 0$

Let $x + 2y = 28$ and $3x + y = 24$

$x + 2y = 28$

x	0	28	4
y	14	0	12

[1]

$3x + y = 24$

x	0	8	4
y	24	0	12

[1]

Maximum value of z is 200 at (4,12)

No. of tennis rackets $x = 4$

No. of cricket bats, $y = 12$

Maximum profit = Rs. 200. [1]

6. Given: Let no. of nuts manufactured = x [1]

And no. of bolts manufactured = y

Maximise profit Z = 17.5x +7y

Subject to the constraints,

$x + 3y \leq 12$

$3x + y \leq 12$

$x \geq 0, \ y \geq 0$

Let $x + 3y = 12$ and $3x + y = 12$ [1]

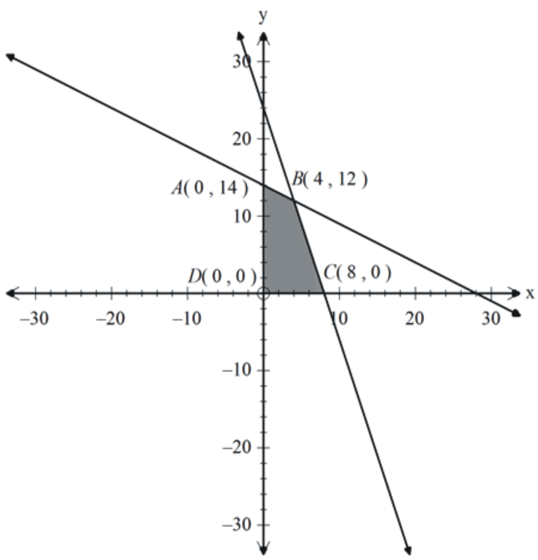
$x + 3y = 12$

x	12	0	3
y	0	4	3

$3x + y = 12$

x	4	0	3
y	0	12	3

[1]



[1]

Corner Points	z = 17.5x + 7y
A(0,4)	$0+7(4)=28$
B(3,3)	$17.5(3)+7(3)=52.5+21=73.5$
C(4,0)	$17.5(4)+7(0)=70+0=70$
D(0,0)	$0+0=0$

[1]

Here the manufacture has to produce 3 units each of nuts and bolts to get the maximum profit of Rs. 73.50. [1]

7. Let number of electronic sewing machines be x.

Let the number of manual sewing machines = y

Maximise profit, $Z = 22x + 18y$

Subject to the constraints,

$x + y \leq 20$

$360x + 240y \leq 5760$

$36x + 24y \leq 576$

$9x + 6y \leq 144$

$3x + 2y \leq 48$

$x \geq 0; y \geq 0$ [1]

Let $x + y = 20$

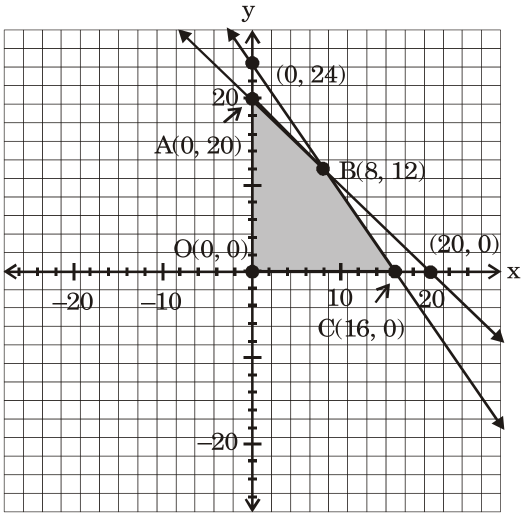
X	20	0	8
Y	0	20	12

[1]

Let $3x + 2y = 48$

X	16	0	8
Y	0	24	12

[1]



[1]

Corner Points	$Z = 22x + 18y$
$A(0,20)$	$Z = 0 + 18(20) = 360$
B(8,12)	$Z = 22(8) + 18(12) = 176 + 216 = 392$
$A(16,0)$ $A(0,0)$	$Z = 22(16) + 0 = 352$ $Z = 0 + 0 = 0$

[1]

Number of electronic sewing machines,

$$x = 8$$

Number of manual sewing machines,

$$y = 12$$

$\therefore$ maximum profit = Rs. 392. [1]

8. Let the factory manufacture x screws of type A and y screws of type B on each day. Therefore, $x \geq 0$ and $y \geq 0$ [1]

The given information can be compiled in a table as follows.

	Screw A	Screw B	Availability
Automatic Machine (min)	4	6	4 × 60 =120
Hand Operated Machine (min)	6	3	4 × 60 =120

[1]

The profit on a package of screws A is Rs 0.7 and on the package of screws B is Rs 1.

Therefore, the constraints are

$4x + 6y \leq 240$

$6x + 3y \leq 240$

Total profit, Z = $0.7x + 1y$

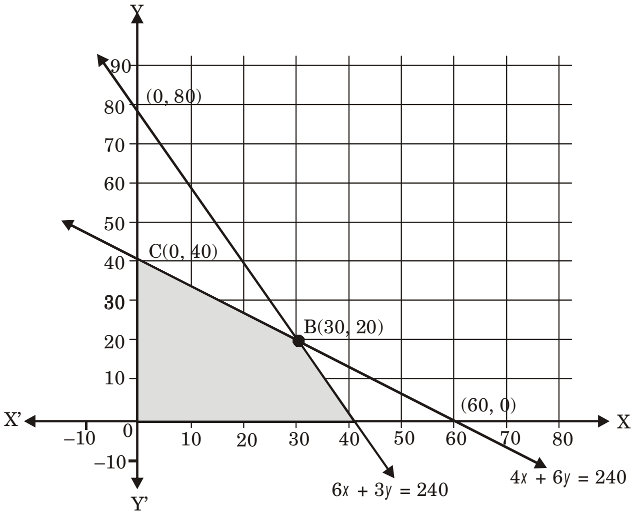
The mathematical formulation of the given problem is Maximize Z = $0.7x + 1y$ ..(1) subject to the constraints,

$4x + 6y \leq 240$ …(2)

$6x + 3y \leq 240$ …(3)

$x, y \geq 0$ …(4) [1]

The feasible region determined by the system of constraints is.



[1]

The corner points are A (40, 0), B (30, 20), and C (0, 40).

The values of Z at these corner points are as follows.

Corner point	Z = 0.7x + 1y	
A(40, 0)	28	
B(30, 20)	41	Maximum
C(0, 40)	40	
D(0, 0)	0	

[1]

The maximum value of Z is 41 at (30, 20).

Thus, the factory should produce 30 packages of screws A and 20 packages of screws B to get the maximum profit of Rs 41. [1]

9. Let object of type A = x

Object of type B = y

If $z_{max} = 100x + 120y$ [1]

	Type A	Type B	
Workers	2	3	30
Capital	3	1	17

[1]

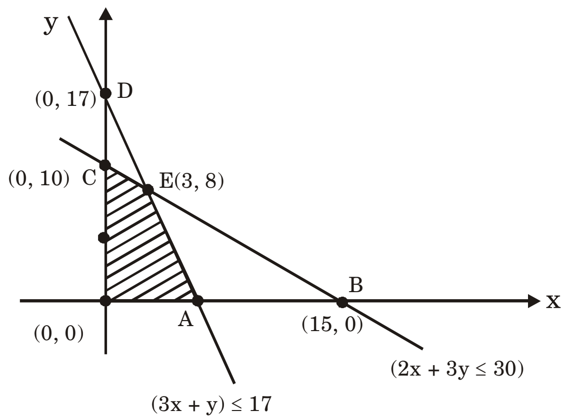
Subject to,

$2x + 3y \leq 30$

$3x + y \leq 17$

$x \geq 0$

$y \geq 0$ [1]



[1]

Points	Coordinate	$z_{max} = 100x + 120y$
O	(0,0)	Z = 0
A	$\left(\frac{17}{3}, 0\right)$	$Z = \frac{1700}{3}$
E	(3,8)	Z = 300 + 960 = 1260
C	(0,10)	Z = 1200

[1]

Maximum revenue = 1260 [1]

10. Let pieces of type A manufactured per week = x

Let pieces of type B manufactured per week = y

Companies profit function which is to be maximized: $Z = 80x + 120y$ [1]

	Fabricating hours	Finishing hours
A	9	1
B	12	3

[1]

Constraints : Maximum number of fabricating hours = 180

$\therefore 9x + 12y \leq 180$

$\Rightarrow 3x + 4y \leq 60$ [1]

Where $9x$ is the fabricating hours spent by type A teaching aids, and $12y$ hours spent on type B. and Maximum number of finishing hours = 30

$\therefore x + 3y \leq 30$

where x is the number of hours spent on finishing aid A while 3y on aid B.

So, the LPP becomes:

Z (MAXIMISE) = 80x + 120y

Subject to $3x + 4y \le 60$ [1]

$x + 3y \le 30$

$x \ge 0$

$y \ge 0$

Solving it Graphically:

$Z = 80x + 120y$ at $(0,15) = 1800$

$Z = 1200$ at $(0,10)$

$Z = 1600$ at $(20,0)$ [1]

$Z = 960 + 720$ at $(12,6) = 1680$

Maximum profit is at (0.15)

Thus, Teaching aid A = 0

Teaching aid B = 15 [1]

11. Let x be the man helpers and y be the woman helpers

Pay roll: $Z = 225\,x + 200\,y$ [1]

Subject to constraints:

$x + y \le 10$

$3x + 4y \ge 34$

$8x + 5y \ge 68$

$x \ge 0, y \ge 0$ [1]

At $A\left(0, \frac{68}{5}\right), Z(A) = Rs.2720$

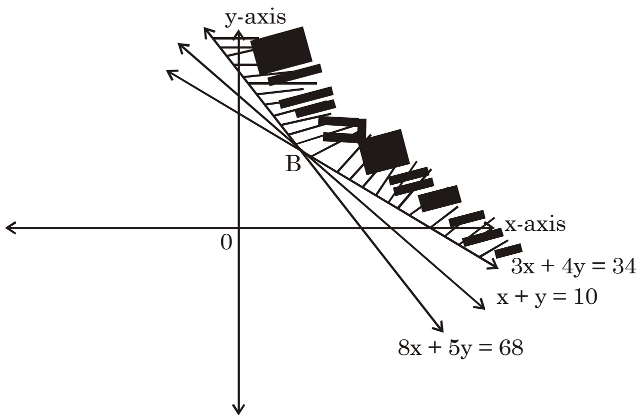
At B $(6, 4)$ Z (B) = Rs. 2150 Minimum [1]

At $C\left(\frac{34}{5}, 0\right) Z(C) = Rs2550$ [1]

Minimum $Z =$ Rs. 2150 at $(6,4)$

[Feasible region is unbounded and to check minimum

Of Z, $225x + 200y < 2150$ corresponding line is outside of the shaded region] [1]



[1]

12. Let Rs.'x' invest in bond A and Rs.'y' invest in bond 'B'

Then A.T.P.

Maximize $z = \frac{10}{100}x + \frac{9}{100}y$ ...(1) [1]

Subject to constraints

$x \ge 20,000$

$x + y \le 50,000$

$y \ge 10,000$ [1]

And $x \ge y$

or $x - y \ge 0$

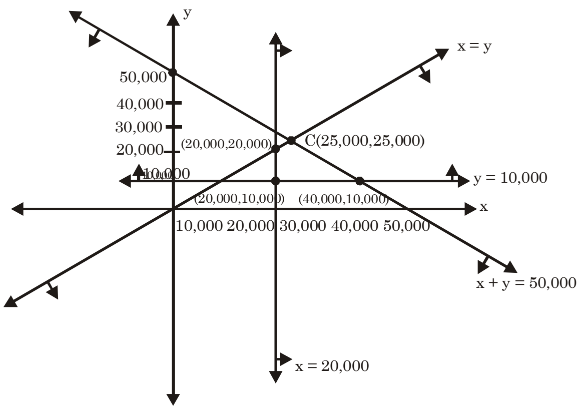
and $x \ge 0, y \ge 0$ [1]

Now change inequalities into equations

$x + y = 50000, x = 20000, y = 10000$ *and* $x = y$

X	0	50 000
Y	50 000	0

[1]



[1]

Table:

Points	$Z = \frac{10}{100}x + \frac{9}{100}y$
A(20 000,10 000)	Z = Rs.2 900
B(40 000,10 000)	Rs.4 900 ← ***Maximize***
C (25 000,25000)	Rs.4 750
D(20 000,20 000)	Rs.3 800

[1]

So he has to invest Rs.40 000 in 'A' and Rs.10 000 in bond 'B' to get maximum return Rs.4 900.

13.

	Tailor A	Tailor B	Mini. total no.
No. of shirts	6	10	60
No. of trousers	4	4	32
wage	Rs. 300/day	Rs. 400/da	

[1]

Let tailor A and tailor B works for X days and Y days respectively [1]

$\therefore x \geq 0, y \geq 0$

Minimum number of shirts = 60

$\therefore 6x + 10y \geq 60$

$3x + 5y \geq 30$ [1]

Minimum no. of trousers = 32

$\therefore 4x + 4y \geq 32$

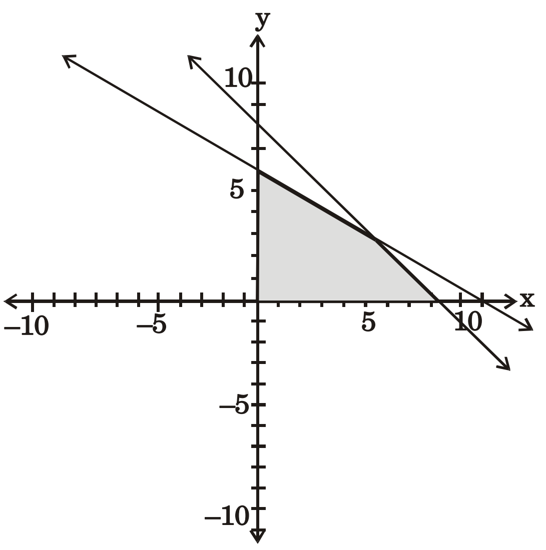
$\Rightarrow x + y \geq 8$

Let Z be the total labour cost

$\therefore z = 300x + 400y$ [1]

$\therefore$ The given L.P reduces to $z = 300x + 400y$

$x \geq 0,\ y \geq 0,\ 3x + 5y \geq 30$ and $x + y \geq 8$ [1]



[1]

14. $2x + 4y \leq 8$

Let $2x + 4y = 8$ or

$x + 2y = 4$ ....(i)

X	0	4
Y	2	0

[1]

$3x + y \leq 6$

Let $3x + y = 6$ ....(ii)

X	0	2
Y	6	0

[1]

$x + y \leq 4$

Let $x + y = 4$

X	0	4
Y	4	0

[1]

For point of intersection, multiplying eqn (i) by 3 and (ii) by 1.

$$\begin{array}{l} 3x + 6y = 12 \\ 3x + y = 6 \\ - \quad - \quad - \\ \hline 5y = 6 \end{array}$$

$\Rightarrow y = \frac{6}{5}$

Putting the value of y in (i),

$x + 2\left(\frac{6}{5}\right) = 4 \quad x = 4 - \frac{12}{5} = \frac{20 - 12}{5} = \frac{8}{5}$

$\therefore$ Point $B\left(\frac{8}{5}, \frac{6}{5}\right)$ [1]

Corner Points	$Z = 2x + 5y$
$O(0,0)$	0
$A(0,2)$	$0 + 10 = 10$
$B\left(\frac{8}{5}, \frac{6}{5}\right)$	$2\left(\frac{8}{5}\right) + 5\left(\frac{6}{5}\right) = \frac{46}{5} = 9.2$
$C(2,0)$	$4 + 0 = 4$

[1]

Maximum value of Z = 10 at $x = 0, y = 2$. [1]

Smart Notes ....

CHAPTER 13

Probability

Chapter Analysis of Previous Three Years' Board Exams

Number of Questions Asked in Exams

Topics	2018		2017		2016	
	Delhi	**All India**	**Delhi**	**All India**	**Delhi**	**All India**
Probability	2, 6 marks	2, 6 marks	4 marks	4 marks	4 marks	4 marks
Mean and Variance	4 marks	4 marks	4 marks	4 marks	6 marks	6 marks
Independent Events			2 marks	2 marks		

[TOPIC 1] Conditional Probability and Independent Events

Summary

- **Probability:**

  Let S be the sample space and E be the event in an experiment.

  Then,

  $$\text{Probability} = P(E) = \frac{\text{Number of favourable event}}{\text{Total number of events}}$$

  $$= \frac{n(E)}{n(S)}$$

  Where, $0 \le n(E) \le n(s)$

  $\Rightarrow 0 \le P(E) \le 1$

  Hence, the probability of the occurrence of an event E is denoted by $P(E)$

  Now, $P(\overline{E}) = 1 - P(E)$ ($P(\overline{E})$ can also be written as $P(E')$)

  - If probability of any event is one, this does not depict the certainty of that event.
  - In similar way if the probability of any event is zero, this does not depict that the event will never occur.
  - **Mutually Exclusive Event:** The two events which cannot occur simultaneously are called mutually exclusive events.

    Let $B = \{1, 2, 3, 4, 5, 6\}$

    X = the event of occurrence of a number greater than $5 = \{6\}$

    Y = the event of occurrence of an even number $= \{2, 4, 6\}$

    Here, events X and Y are not mutually exclusive because they can occur together when the number 6 comes up.
  - Independent Events: If the occurrence or non-occurrence of one event is unaffected by the occurrence or non-occurrence of other, these events are called independent events.

    Consider an example of drawing two marbles one by one with replacement from a jar containing 2 red marbles and 1 yellow marble

    Now assume, X = the event of occurrence of a red marble in first draw

    And Y = the event of occurrence of a yellow marble in second draw

    So, here the probability of occurrence Y is not affected by that of X.

    Hence, events X and Y are independent events.
  - **Exhaustive Events:** If the performance of random experiment always results in the occurrence of at least one of the given set of events, the set of those events will be known as exhaustive.
  - If their union is the total sample space
  - If event A, B and C are disjoint pairs i.e.,

    Consider an example of throwing a die, $A = \{1, 2, 3, 4, 5, 6\}$

    Now assume X = the event of occurrence of an multiple of $2 = \{2, 4, 6\}$

    Y = the event of occurrence of the number not divisible by $2 = \{1, 3, 5\}$

    Z = the event of occurrence of multiple of $3 = \{3,6\}$

    Here X and Y are mutually exclusive but Y and Z are not.

- **Conditional Probability:**

  The probability of occurrence of event A when B has already been occurred is known as Conditional probability also called probability of occurrence of A w.r.t B.

  Some important formulae related to conditional probability

  - $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, $B \ne \phi$ *i.e.,* $P(B) \ne 0$

- $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$, $A \neq \phi$ *i.e.,* $P(A) \neq 0$
- $P(\overline{A} \mid B) = \dfrac{P(\overline{A} \cap B)}{P(B)}$, $P(B) \neq 0$
- $P(A \mid \overline{B}) = \dfrac{P(A \cap \overline{B})}{P(\overline{B})}$, $P(\overline{B}) \neq 0$
- $P(\overline{A} \mid \overline{B}) = \dfrac{P(\overline{A} \cap \overline{B})}{P(\overline{B})}$, $P(\overline{B}) \neq 0$
- $P(A \mid B) + P(\overline{A} \mid B) = 1$

Some formulae

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ *i.e.,* $\mathrm{P}(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) - P(A \cap B \cap C)$
- $P(\overline{A} \cap B) = P(\text{only } B) = P(B - A) = P(B \text{ but not } A) = P(B) - P(A \cap B)$
- $P(A \cap \overline{B}) = P(\text{only } A) = P(A - B) = P(\text{A but not } B) = P(A) - P(A \cap B)$
- $P(\overline{A} \cap \overline{B}) = P(B - A) = P(\text{neither } A \text{ nor } B) = 1 - P(A \cup B)$

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

2 Marks Questions

1. A black and a red die are rolled together. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

[DELHI 2018]

2. Prove that if E and F are independent events, then the events E and F ' are also independent.

[DELHI 2017]

3. A die, whose faces are marked 1, 2, 3 in red and 4, 5, 6 in green, is tossed. Let A be the event "number obtained is even" and B be the event "number obtained is red". Find if A and B are independent events.

[ALL INDIA 2017]

4 Marks Questions

4. Find the mean number of heads in three tosses of a fair coin

[ALL INDIA 2011]

5. A and B throw a pair of dice alternately, till one of them gets a total of 10 and wins the game. Find their respective probabilities of winning, if A starts first.

[ALL INDIA 2016]

6. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls? Given that

(i) the youngest is a girl

(ii) at least one is a girl.

[DELHI 2014]

7. Probabilities of solving a specific problem independently by A and B are $\frac{1}{2}$ *and* $\frac{1}{3}$ respectively if both try to solve the problem independently, find the probability that (i) the problem is solved (2) exactly one of them solves the problem.

[DELHI 2011]

8. A speaks truth in 60% of the cases, while B in 90% of the cases. In what percent of cases are they likely to contradict other in stating the same fact? In the cases of contradiction do you think the statement of B will carry more weight as he speaks truth in more of cases than A?

[2013]

6 Marks Question

9. Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find the probability distribution of the random variable X, and hence find the mean of the distribution.

[ALL INDIA 2014]

Solutions

1. Let P = The sum is 8.

And Q = Red die resulted in a number less than 4.

Total number of outcomes when two die are thrown is = 36

$$P=\{(2,6),(3,5),(4,4),(5,3),(6,2)\}$$

$$Q=\begin{Bmatrix}(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\(3,1),(3,1),(3,3),(3,4),(3,5),(3,6)\end{Bmatrix}$$

$$\Rightarrow P\cap Q=\{(2,6),(3,5)\}$$ [1]

Now, $P(Q)=\frac{18}{36}=\frac{1}{2}$ *and* $P(P\cap Q)=\frac{2}{36}=\frac{1}{18}$

Required probability

$$P\left(\frac{P}{Q}\right)=\frac{P(P\cap Q)}{P(Q)}$$

$$P\left(\frac{P}{Q}\right)=\frac{\frac{1}{18}}{\frac{1}{2}}$$

$$=\frac{1}{18}\times\frac{2}{1}$$

$$=\frac{1}{9}$$ [1]

2. Given E and F are independent events.

$$\Rightarrow P(E\cap F)=P(E)\times P(F) \qquad (1)$$

To show E and F ' are independent.

$$\Rightarrow P(E\cap F')=P(E)\times P(F')$$

Proceeding from left side we get,

$$=P(E\cap F')$$

$$=P(E)-P(E\cap F)$$ [1]

Using (1) we get,

$$=P(E)-P(E)\times P(F)$$

$$=P(E)[1-P(F)]$$

Using $1-P(F)=P(F')$

$$=P(E)\times P(F')$$ [1]

3. $A=\{2,4,6\}$

$$P(A)=\frac{3}{6}=\frac{1}{2}$$

$$B=\{1,2,3\}$$

$$P(B)=\frac{3}{6}=\frac{1}{2}$$

$$A\cap B=\{2\}$$

$$P(A\cap B)=\frac{1}{6}$$ [1]

Here, $P(A)P(B)=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$

Since $P(A\cap B)\neq P(A)P(B)$

Hence, A and B are not independent events. [1]

4. Let X = the success of getting heads

$$S=\{HHH,HHT,HTH,HTT,THH,THT,TTH,TTT\}$$

X can take 4 values = 0,1,2,3 [1]

Now,

$$P(X=0)=P(TTT)$$

$$P(T)\times P(T)\times P(T)=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$$

$$=P(X=1)=P(HTT)+P(TTH)+P(THT)$$

$$=\left(\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\right)+\left(\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\right)+\left(\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\right)=\frac{3}{8}$$

$P(X=2) = P(HHT) + P(HTH) + P(THH)$

$= \left(\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\right) + \left(\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\right) + \left(\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\right) = \frac{3}{8}$

$P(X=3) = P(HHH)$

$= P(H)\times P(H)\times P(H) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{8}$ [1]

Hence, required probability distribution is:

X	0	1	2	3
P(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

[1]

Mean of $XP(X), \mu = \sum X_i P(X_i)$

$= 0\times\frac{1}{8} + 1\times\frac{3}{8} + 2\times\frac{3}{8} + 3\times\frac{1}{8}$

$= \frac{3}{8} + \frac{3}{4} + \frac{3}{8}$

$= \frac{3}{2}$ [1]

5. Total of 10 : (6, 4) (4, 6) (5, 5)

$\Rightarrow P = \frac{3}{36} = \frac{1}{12}$ [1]

$P(A$ wins)

$= \frac{1}{12} + \frac{11}{12}.\frac{11}{12}.\frac{1}{12} + \frac{11}{12}.\frac{11}{12}.\frac{11}{12}.\frac{11}{12}.\frac{1}{12} + \ldots\ldots$

$\Rightarrow \frac{1}{12}\left(1 + \left(\frac{11}{12}\right)^2 + \left(\frac{11}{12}\right)^4 + \ldots\right)$

$\Rightarrow \frac{1}{12}\left(\frac{1}{1-\left(\frac{11}{12}\right)^2}\right) = \frac{1}{12}.\frac{144}{144-121}$ [1]

$P(A \text{ wins}) = \frac{12}{23}$ [1]

$P(B \text{ wins}) = 1 - P(A \text{ wins}) = 1 - \frac{12}{23} = \frac{11}{23}$ [1]

6. Let the event be

A: both the children are girls

B: the youngest is a girl

C: at least one is a girl

S (*Sample space*) $= \{bb, bg, gb, gg\}$

$A = \{gg\}$

(i) $B = \{bg, gg\} \therefore P(\text{B}) = \frac{2}{4}$

$A\cap B = \{gg\} \quad \therefore P(A\cap B) = \frac{1}{4}$ [1]

$P\left(\frac{A}{B}\right) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}$ [1]

(ii) $C = \{bg, \text{gb}, gg\} \therefore P(\text{C}) = \frac{3}{4}$

$A\cap C = \{gg\} \quad \therefore P(A\cap C) = \frac{1}{4}$ [1]

$P\left(\frac{A}{C}\right) = \frac{P(A\cap C)}{P(\text{C})} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$ [1]

7. Given Let p_1 be the probability of solving a specific problem by A

Let p_2 be the probability of solving a specific problem by B

q_1 be the probability of not solving a specific problem by A.

q_2 be the probability of not solving a specific problem by B. [1]

Here $p_1 = \frac{1}{2}$

$q_1 = 1 - \frac{1}{2} = \frac{1}{2}$

$p_2 = \frac{1}{3}$

$q_2 = 1 - \frac{1}{3} = \frac{2}{3}$ [1]

(1) P (problem is solved)

$= 1 -$ P (Problem is not solved)

$= 1 - q_1 q_2$

$= 1 - \frac{1}{2}\times\frac{2}{3}$

$= 1 - \frac{1}{3} = \frac{2}{3}$ [1]

(2) P (problem one of them is solved)

$= p_1 q_2 + q_1 q_2$

$= \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{1}{3}$

$= \frac{2}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$ [1]

8. Given : Let $p_1 = p$ (A speaks the truth)

$60\% = \frac{60}{100} = \frac{6}{10}$

$q_1 = 1 - p_1 = 1 - \frac{6}{10} = \frac{4}{10}$ (A doesn't speak the truth) [1]

p_2 = p(B speaks the truth) $= 90\% = \frac{90}{100} = \frac{9}{10}$

$q_2 = 1 - p_2 = 1 - \frac{9}{10} = \frac{1}{10}$ (B doesn't speak the truth)

p(contradiction) = $p_1 q_2 + q_1 p_2$ [1]

$= \frac{6}{10} \times \frac{1}{10} + \frac{4}{10} \times \frac{9}{10}$

$= \frac{6+36}{100} = \frac{42}{100} = 0.42$ [1]

Required % $= \frac{42}{100} \times 100 = 42\%$

A and B are likely to contradict each other in 42% of the cases. [1]

9. First six numbers are 1, 2, 3, 4, 5, 6.

X is bigger number among 2 number so

Variable (X)	2	3	4	5	6
Probability P(X)	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

[1]

If X = 2

for P(X) = Prob. of event that bigger of the 2 chosen number is 2

So, Cases = (1, 2)

So, $P(X) = \frac{1}{{}^6C_2} = \frac{1}{15}$ (1) [1]

If $X = 3$

So, favourable cases are $(1,3),(2,3)$

$P(X) = \frac{2}{{}^6C_2} = \frac{2}{15}$ (2)

If $X = 4$

So, favourable cases are $(1,4),(2,4),(3,4)$ [1]

$P(X) = \frac{3}{{}^6C_3} = \frac{3}{15}$ (3)

If $X = 5$

So, favourable cases are $(1,5),(2,5),(3,5),(4,5)$

$P(X) = \frac{4}{15}$ (4) [1]

If $X = 6$

So, favourable cases are

$(1,6),(2,6),(3,6),(4,6),(5,6)$

$P(X) = \frac{5}{15}$ (5)

We can put all value of P(X) in chart, So

Variable (X)	2	3	4	5	6
Probability P(X)	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

[1]

and required mean

$= 2\left(\frac{1}{15}\right) + 3\left(\frac{2}{15}\right) + 4\left(\frac{3}{15}\right) + 5\left(\frac{4}{15}\right) + 6\left(\frac{5}{15}\right)$

$= \frac{70}{15} = \frac{14}{3}$ [1]

[TOPIC 2] Baye's Theorem and Probability Distribution

Summary

- **BAYES' theorem:**
  - If E_1, E_2, E_3 .... E_n are n non-empty constituting a partition of sample space S i.e., S_1, S_2, S_3 .... S_n are pair wise disjoint and $E_1 \cup E_2 \cup E_3 \cup ..... \cup E_n = S$ and A is any event of non- zero probability, then

$$P(E \mid A) = \frac{P(E_i).P(A \mid E_i)}{\sum_{j=1}^{n} P(E_j) P(A \mid E_j)}, i = 1,2,3,.......n$$

  - For example,

$$P(E_1 \mid A) = \frac{P(E_1).P(A \mid E_1)}{P(E_1).P(A \mid E_1) + P(E_2).P(A \mid E_2) + P(E_3).P(A \mid E_3)},$$

$i = 1,2,3,.......n$

  - It is also known as the formula for the probability of cause.
  - Prior probabilities are the probabilities which are known before the experiment takes place.
  - $P(A \mid E_n)$ are called posterior probabilities.
- **Random Variable:**

  A real valued function defined over the sample space of an experiment is known as random variable. It is denoted by uppercase letters X, Y, Z etc.

  - **Discrete random variable :** When only finite or countably infinite number of values can be taken by the random variable then it is called discrete random variable.
  - **Continuous random variable:** When any value between two given limits can be taken by the variable then it is called continuous random variable.

    If the values of a random variable together with the corresponding probability are known, then this is called the **probability distribution of the random variable**.
- **Formulae:**
  - Mean or Expectation of a random variable

    $X = X = \mu = \sum_{i=1}^{n} x_i P_i$

  - Variance = $(\sigma^2) = \sum_{i=1}^{n} P_i x^2{}_i - \mu^2$
  - Standard deviation = $\sigma = \sqrt{\text{Variance}}$
- **Bernoulli Trials:**

  They are basically known as trials of a random experiment.

  If they satisfy the following conditions:

  - There should be a finite number of trials.
  - The trials should be independent.
  - Each trial has exactly two outcomes: success or failure.
  - The probability of success remains the same in each trial
- **Binomial distribution:**

  A Binomial distribution with probability of success in each trial as p and with n Bernoulli trials is denoted by $B(n, p)$

  n and p are the parameters of Binomial Distribution

  Therefore the expression $P(x = r)$ or $P(r)$ is called the probability function of Binomial Distribution.

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 2

4 Marks Questions

1. An experiment succeeds thrice as often as it fails. Find the probability that in the next five trials, there will be at least 3 successes.

[ALL INDIA 2014]

2. There are 4 cards numbered 1, 3, 5 and 7, one number on one card. Two cards are drawn at random without replacement, Let X denote the sum of the numbers on the two drawn cards. Find the mean and variance of X.

[ALL INDIA 2017]

3. A and B throw a die alternatively till one of them gets a number greater than four and wins the game. If A starts the game, what is the probability of B winning?

[ALL INDIA 2015]

4. A die is thrown three times. Events A and B are defined as below :

   A : 5 on the first and 6 on the second throw.

   B : 3 or 4 on the third throw.

   Find the probability of B, given that A has already occurred.

[ALL INDIA 2015]

5. 40% students of a college reside in hostel and the remaining reside outside. At the end of the year, 50% of the hostellers got A grade while from outside students, only 30% got A grade in the examination. At the end of the year, a student of the college was chosen at random and was found to have gotten A grade. What is the probability that the selected student was a hosteller?

[ALL INDIA 2015]

6. Of the students in a school, it is known that 30% have 100% attendance and 70% students are irregular. Previous year results report that 70% of all students who have 100% attendance attain A grade and 10% irregular students attain A grade in their annual examination. At the end of the year, one student is chosen at random from the school and he was found to have an A grade. What is the probability that the student has 100% attendance? Is regularity required only in school? Justify your answer.

[ALL INDIA 2017]

7. A bag X contains 4 white balls and 2 black balls, while another bag Y contains 3 white balls and 3 black balls. Two balls are drawn (without replacement) at random from one of the bags and were found to be one white and one black. Find the probability that the balls were drawn from bag Y.

[ALL INDIA 2016]

8. In a hockey match, both teams A and B scored same number of goals up to the end of the game, so to decide the winner, the referee asked both the captains to throw a die alternately and decided that the team, whose captain gets a six first, will be declared the winner. If the captain of team A was asked to start, find their respective probabilities of winning the match and state whether the decision of the referee was fair or not.

[ALL INDIA 2013]

9. How many times must a man toss a fair coin, so that the probability of having at least one head is more than 80%?

[ALL INDIA 2012]

10. The random variable X can take only the values 0, 1, 2, 3. Given that $P(X=0)=P(X=1)=p$ and $P(X=2)=P(X=3)$ such that $\sum P_i X_i^2 = 2\sum P_i X_i$, find the value of p.

[DELHI 2017]

11. A and B throw a pair of dice alternately. A wins the game if he gets a total of 7 and B wins the game if he gets a total of 10. If A starts the game, then find the probability that B wins.

[DELHI 2016]

12. A bag A contains 4 black and 6 red balls and bag B contains 7 black and 3 red balls. A die is thrown. If 1 or 2 appears on it, then bag A is chosen, otherwise bag B. If two balls are drawn at random (without replacement) from the selected bag, find the probability of one of them being red and another black.

[DELHI 2015]

6 Marks Questions

13. Three numbers are selected at random (without replacement) from first six positive integers. Let X denote the largest of the three numbers obtained. Find the probability distribution of X. Also, find the mean and variance of the distribution.

[ALL INDIA 2016]

14. Suppose a girl throws a die. If she gets a 1 or 2, she tosses a coin three times and notes the number of heads. If she gets 3, 4, 5 or 6, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one tail, what is the probability that she threw 3, 4, 5 or 6 with the die?

[DELHI 2018]

15. Two numbers are selected at random (without replacement) from the first five positive integers, Let X denote the larger of the two numbers obtained. Find the mean and variance of X.

[DELHI 2018]

16. An urn contains 3 white and 6 red balls. Four balls are drawn one by one with replacement from the urn. Find the probability distribution of the number of red balls drawn. Also find mean and variance of the distribution.

[DELHI 2016]

17. Of the students in a college ,it is known that 60% reside in hostel and 40% are day scholar (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade in their annual examination. At the end of the year one student is chosen at random from the collage and he has an A grade, what is the probability that the student is a hostlier?

[ALL INDIA 2012]

18. Suppose 5%of men and 0.25%of women have grey hair. A grey haired person is selected at random. What is the probability of this person being male? Assume that are equal number of males and females.

[DELHI 2011]

19. There are three coins. One is a two-headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the times and third is also a biased coin that comes up tails 40% of the times. One of the three coins is chosen at random and tossed, and it shows heads. What is the probability that it was the two-headed coin.

[ALL INDIA 2014]

20. Assume that the chances of a patient having a heart attack is 40%. Assuming that a meditation and yoga course reduces the risk of heart attack by 30% and prescription of drug reduces its chance by 25%. At a time a patient can choose any one of the two option with equal probabilities. It is given that after going through one of the two options, the patient selected at random suffers a heart attack. Find the probability that the followed a course of meditation and yoga.

[2013]

Solutions

1. In Binomial distribution

$$(p+q)^n = {}^nC_0 . p^n + {}^nC_1 . p^{n-1}.q^1 + {}^nC_2 . p^{n-2}.q^2 + \ldots\ldots + {}^nC_n.q^n \quad [1]$$

if p = probability of success

q = prob. of fail

given that $p = 3q$ ...(1)

we know that $p + q = 1$

so, $3q + q = 1$

$$q = \frac{1}{4}$$

$$\text{so, } p = \frac{3}{4} \quad [1]$$

Now given ⇒ $n = 5$ we required minimum 3 success

$$(p+q)^5 = {}^5C_0 . p^5 + {}^5C_1 . p^{5-1}.q^1 + {}^5C_2 . p^{5-2}.q^2$$

$$= {}^5C_0 . \left(\frac{3}{4}\right)^5 + {}^5C_1 . \left(\frac{3}{4}\right)^4 . \left(\frac{1}{4}\right)^1 + {}^5C_2 . \left(\frac{3}{4}\right)^3 . \left(\frac{1}{4}\right)^2 \quad [1]$$

$$= \frac{3^5}{4^5} + \frac{5.3^4}{4^5} + \frac{10.3^3}{4^5}$$

$$= \frac{(3^5 + 5.3^4 + 10.3^3)}{4^5}$$

$$= \frac{3^3[9+15+10]}{4^5} \quad [1]$$

2. X denote sum of the numbers so, X can be 4, 6, 8, 10, 12

X	Number on card	$P(x)$	$X\,P(x)$	$X^2\,P(x)$
4	(1, 3)	$\frac{1}{4}\times\frac{1}{3}\times 2=\frac{1}{6}$	$\frac{2}{3}$	$\frac{8}{3}$
6	(1, 5)	$\frac{1}{4}\times\frac{1}{3}\times 2=\frac{1}{6}$	1	6
8	(3, 5) or (1, 7)	$\frac{1}{4}\times\frac{1}{3}\times 2+\frac{1}{4}\times\frac{1}{3}\times 2=\frac{1}{3}$	$\frac{8}{3}$	$\frac{64}{3}$
10	(3, 7)	$\frac{1}{4}\times\frac{1}{3}\times 2=\frac{1}{6}$	$\frac{5}{3}$	$\frac{50}{3}$
12	(5, 7)	$\frac{1}{4}\times\frac{1}{3}\times 2=\frac{1}{6}$	2	24

[2]

Mean $=\sum X P(x)=8$

Variance

$$=\sum X^2 P(x)-\left(\sum X P(x)\right)^2=\frac{212}{3}-64=\frac{20}{3}$$ [2]

3. Let E be the event of getting number greater than 4

$$\therefore P(E)=\frac{1}{3} \text{ and } P(\overline{E})=\frac{2}{3}$$

Required Probability

$$= P\left(\overline{E}E \text{ or } \overline{E}\overline{E}\overline{E}E \text{ or } \overline{E}\overline{E}\overline{E}\overline{E}\overline{E}E \text{ or} \ldots\right)$$ [2]

$$=\frac{2}{3}\cdot\frac{1}{3}+\left(\frac{2}{3}\right)^3\cdot\frac{1}{3}+\left(\frac{2}{3}\right)^5\cdot\frac{1}{3}+\ldots\ldots\infty$$

$$=\frac{2}{9}[1+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^4+\ldots\ldots\infty]$$

$$=\frac{2}{9}\times\frac{9}{5}=\frac{2}{5}$$ [2]

4. Events of

$$A=\{(5,6,1),(5,6,2),(5,6,3),(5,6,4),(5,6,5),(5,6,6)\}$$

$$P(A)=\frac{6}{6\times6\times6}=\frac{1}{36}$$ [1]

$P(B)=P$(getting 3 or 4 on the third throw) [1]

$$A\cap B=\{(5,6,\ 3),(5,6,4)\}$$

$$\Rightarrow P(A\cap B)=\frac{2}{6\times6\times6}=\frac{1}{108}$$ [1]

$$P(B|A)=\frac{P(A\cap B)}{P(A)}=\frac{1}{3}$$ [1]

5. Let E_1, E_2 and E_3 be the events such that

E_1: students residing in hostel

E_2 : students residing outside the hostel

E_3 : students getting 'A' grade [1]

$$P(E_1)=\frac{40}{100},\ p(E|E_1)=\frac{50}{100}$$

$$P(E_2)=\frac{60}{100},\ p(E|E_2)=\frac{30}{100}$$ [1]

$$P(E_1|E)=\frac{P(E_1).P(E/E_1)}{P(E_1).P(E/E_1)+P(E_2)P(E/E_2)}$$ [1]

$$=\frac{\frac{40}{100}\times\frac{50}{100}}{\frac{40}{400}\times\frac{50}{100}+\frac{30}{100}\times\frac{60}{100}}$$

$$=\frac{10}{19}$$ [1]

6. Let E_1 be students having 100% attendance
E_2 be students having irregular attendance
E be students having A grade

$$P(E_1)=\frac{30}{100}\ P(E_2)=\frac{70}{100}$$ [1]

$$P\left(\frac{E}{E_1}\right)=\frac{70}{100}\times\frac{30}{100}=21\%$$

$$P\left(\frac{E}{E_2}\right)=\frac{10}{100}\times\frac{70}{100}=7\%$$ [1]

By Baye's theorem,

$$P\left(\frac{E_1}{E}\right)=\frac{P(E_1)P\left(\frac{E}{E_1}\right)}{P(E_1)P\left(\frac{E}{E_1}\right)+P(E_2)P\left(\frac{E}{E_2}\right)}$$ [1]

$$=\frac{\frac{30}{100}\times\frac{21}{100}}{\frac{30}{100}\times\frac{21}{100}+\frac{70}{100}\times\frac{7}{100}}$$

$$=\frac{63}{63+49}=\frac{63}{112}$$ [1]

7. Bag X contain 4 White and 2 Black balls
Bag Y contain 3 White and 3 Black balls
Event $E_1 \rightarrow$ Bag X is selected
Event $E_2 \rightarrow$ Bag Y is selected
Event $A \rightarrow$ 1 white 1 black is taken out [1]

$$P(E_1)=\frac{1}{2}, P(E_2)=\frac{1}{2}$$

$$=P\left(\frac{A}{E_1}\right)=\frac{{}^4C_1\times{}^2C_1}{{}^6C_2}$$

$$=P\left(\frac{A}{E_2}\right)=\frac{{}^3C_1\times{}^3C_1}{{}^6C_2}$$ [1]

$$P(A) = P(E1)\ P(A|E1) + P(E2)\ P(A|E2)$$

$$= \frac{1}{2} \times \frac{{}^4C_1\ {}^2C_1}{{}^6C_2} + \frac{1}{2} \times \frac{{}^3C_1\ {}^3C_1}{{}^6C_2}$$ [1]

Probability balls are drawn from bag

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2)\,P\left(\frac{A}{E_2}\right)}{P(A)}$$

$$\Rightarrow \frac{9}{17}$$ [1]

8. $P(6\,\text{get}) = \frac{1}{6}$

$$P(6\,\text{not get}) = P(\overline{6\,\text{get}}) = \frac{5}{6}$$

$$P(\text{A win}) = P(A\,\text{get}\,6) + P(\overline{6\,\text{get}}).P(\overline{6\,\text{get}}).$$

$$P(6\,\text{get}) + P(\overline{6\,\text{get}}).P(\overline{6\,\text{get}}).$$

$$P(\overline{6\,\text{get}}).P(\overline{6\,\text{get}}).P(6\,\text{get}) + \ldots\ldots\ldots\ldots + \infty$$ [1]

$$P(\text{A win}) = \frac{1}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} + \ldots\infty$$

$$= \frac{1}{6} + \left(\frac{5}{6}\right)^2 \times \frac{1}{6} + \left(\frac{5}{6}\right)^4 \times \frac{1}{6} + \ldots\ldots + \infty$$ [1]

$$\because S_\infty = \frac{a}{1-r}$$

$$= \frac{\frac{1}{6}}{\left(1 - \frac{25}{36}\right)} = \frac{36}{11\,X 6} = \frac{6}{11}$$ [1]

Similarly winning for B

P(B win) = 1 - P(A win)

$$= 1 - \frac{6}{11} = \frac{5}{11}$$ [1]

9. Let no. of times of tossing a coin be n.

Here, Probability of getting a head in a chance

$$= p = \frac{1}{2}$$ [1]

Probability of getting no head in a chance

$$= q = 1 - \frac{1}{2} = \frac{1}{2}$$

Now, P (having at least one head)

$$= P(X \geq 1)$$

$$= 1 - P(X = 0)$$

$$= 1 - n_{C_0}\ p^0.q^{n-0}$$

$$= 1 - 1 \times 1\left(\frac{1}{2}\right)^n$$ [1]

From question

$$1 - \left(\frac{1}{2}\right)^n > \frac{80}{100}$$

$$\Rightarrow 1 - \left(\frac{1}{2}\right)^n > \frac{8}{10} \Rightarrow 1 - \frac{8}{10} > \frac{1}{2^n}$$ [1]

$$\Rightarrow \frac{1}{5} > \frac{1}{2^n} \Rightarrow 2^n > 5$$

$$\Rightarrow n \geq 3$$ [1]

A man must have to toss a fair coin 3 times.

10. $P(x=0) = P(x=1) = p$ and $P(x=2) = P(x=3)$

$$P(0) + P(1) + P(2) + P(3) = 1$$

$$p + p + P(3) + P(3) = 1$$

$$2P(3) = 1 - 2p$$

$$P(3) = \frac{1-2p}{2}$$ [1]

X_i	P_i	P_iX_i	$P_iX_i^2$
0	p	0	0
1	p	p	p
2	$\frac{(1-2p)}{2}$	$(1-2p)$	$2(1-2p)$
3	$\frac{(1-2p)}{2}$	$\frac{3(1-2p)}{2}$	$\frac{9(1-2p)}{2}$

[1]

We have $\sum P_iX_i^2 = 2\sum P_iX_i$

$$p+2(1-2p)+\frac{9}{2}(1-2p)$$

$$=2\left[p+(1-2p)+\frac{3(1-2p)}{2}\right]$$

$$=\frac{2p+4-8p+9-18p}{2}+2\left(\frac{2p+2-4p+3-6p}{2}\right)$$ [1]

$$=-24p+13=2(5-8p)$$

$$=-24p+13=10-16p$$

$$=-13-10=-16p+24p$$

$$=8p=3\Rightarrow p=\frac{3}{8}$$ [1]

11. Two dice can be thrown ways.

'A total of 7' can be obtained as

(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3) i.e., in 6 ways.

Let $p_1 = P(\text{sum } 7) = \frac{6}{36}=\frac{1}{6}$

$\therefore q_1 = 1-\frac{1}{6}=\frac{5}{6}$ [1]

'A total of 10' can be obtained as

(4, 6), (6, 4), (5, 5) i.e., in 3 ways

Let $p_2 = P(\text{sum } 10) = \frac{3}{36}=\frac{1}{12}$

$q_2 = 1-\frac{1}{12}=\frac{11}{12}$ [1]

B can win in the 2nd, 4th and 6th throw.

$P(B \text{ wins})$

$$= q_1p_2+q_1q_2q_1p_2+(q_1q_2)^2q_1p_2+\ldots\ldots\infty$$

$$=\frac{5}{6}\times\frac{1}{12}+\frac{5}{6}\times\frac{11}{12}+\frac{5}{6}\times$$

$$\frac{1}{12}+\left(\frac{5}{6}\times\frac{11}{12}\right)^2\times\frac{5}{6}\times\frac{1}{12}+\ldots.\infty$$ [1]

Here 'a'= $\frac{5}{72}$, $S_\infty = \frac{a}{1-r}$, r = $\frac{55}{72}$

$$=\frac{\frac{5}{72}}{1-\frac{55}{72}}$$

$$=\frac{\frac{5}{72}}{\frac{72-55}{72}}=\frac{5}{72}\times\frac{72}{17}=\frac{5}{17}$$ [1]

12. Let event E_1 = The die shows 1 0r 2.

Let event E_2 = The die shows 3,4,5 or 6

Let event A = Getting one red and one black ball.

$$\therefore P(E_1)=\frac{2}{6}=\frac{1}{3}\quad, P(E_2)=\frac{4}{6}=\frac{2}{3}$$

$P\left(\frac{A}{E_1}\right)=P$ (If E_1 occurs, drawing 1 red, 1 black ball from bag A)

$$=\frac{{}^4C_1\times{}^6C_1}{{}^{10}C_2}$$

$$=\frac{\frac{4}{1}\times\frac{6}{1}}{\frac{10.9}{2.1}}=\frac{4\times6\times2}{10\times9}=\frac{8}{15}$$ [1]

(If E_2 occurs, drawing 1 red, 1 black ball from bag B)

$P\left(\frac{A}{E_2}\right)=P$ [1]

$$=\frac{{}^7C_1\times{}^3C_1}{{}^{10}C_2}=\frac{\frac{7}{1}\times\frac{3}{1}}{\frac{10.9}{2.1}}=\frac{7\times3\times2}{10\times9}=\frac{7}{15}$$ [1]

By the law of total probability.

$$P(A)=P(E_1)\times P\left(\frac{A}{E_1}\right)+P(E_2)\times P\left(\frac{A}{E_2}\right)$$

$$=\frac{1}{3}\times\frac{8}{15}+\frac{2}{3}\times\frac{7}{15}=\frac{8+14}{45}=\frac{22}{45}$$ [1]

13. First six positive integers are 1, 2, 3, 4, 5, 6

Three numbers are selected at random without replacement

So, total number of ways ${}^6C_3=20$ [1]

Let, x denote the larger of three numbers

So x can take values 3, 4, 5, 6

$$p(x=3)=\frac{1}{20}$$

$$p(x=4)=\frac{3}{20}$$

$$p(x=5)=\frac{6}{20}$$ [1]

$$p(x=6)=\frac{10}{20}$$

X	3	4	5	6
$P(x)$	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{6}{20}$	$\frac{10}{20}$
$Xp(x)$	$\frac{3}{20}$	$\frac{12}{20}$	$\frac{30}{20}$	$\frac{60}{20}$
$x^2 p(x)$	$\frac{9}{20}$	$\frac{48}{20}$	$\frac{150}{20}$	$\frac{360}{20}$

[2]

$$Mean = \sum \times p(x)$$

$$= \frac{3}{20} + \frac{12}{20} + \frac{30}{20} + \frac{60}{20}$$

$$= \frac{105}{20} = 5.25$$ [1]

Variance = $\sum x^2\ p(x) - \left(\sum x(x)\right)^2$

$$= \left(\frac{567}{20}\right) - \left(\frac{105}{20}\right)^2 = 28.35 - 27.56 = 0.79$$ [1]

14. Let E_1 be the event that the outcome on the die is 1 or 2 and E_2 be the event that the outcome on the die is 3,4,5, or 6.

$$\therefore P(E_1) = \frac{2}{6} = \frac{1}{3} \text{ and } P(E_2) = \frac{4}{6} = \frac{2}{3}$$ [1]

Let A be the event of getting exactly one head.

$P(A \mid E_1)$ = Probability of getting exactly one tail by tossing the coin three times if she gets 1 or 2

$$= \frac{3}{8}$$ [1]

$P(A \mid E_2)$ = Probability of getting exactly one tail in a single throw of coin if she gets 3, 4, 5, or 6

$$= \frac{1}{2}$$ [1]

The probability that the girl threw 3, 4, 5, or 6 with the die, If she obtains exactly one tail, is given by $P(E_2 \mid A)$

By using Bayes' theorem, we obtain

$$P(E_2 \mid A) = \frac{P(E_2).P(A \mid E_2)}{P(E_1).P(A \mid E_1) + P(E_2).P(A \mid E_2)}$$ [1]

$$= \frac{\frac{2}{3} \times \frac{1}{2}}{\frac{1}{3} \times \frac{3}{8} + \frac{2}{3} \times \frac{1}{2}}$$

$$= \frac{\frac{1}{3}}{\frac{1}{3}\left(\frac{3}{8} + 1\right)}$$ [1]

$$= \frac{1}{\frac{11}{8}}$$

$$= \frac{8}{11}$$ [1]

15. Given that two numbers are selected from 1,2,3,4,5.

Total ways are $5 \times 4 = 20$ Sample space

$$S = \begin{Bmatrix} (1,2),(1,3),(1,4),(1,5) \\ (2,1),(2,3),(2,4),(2,5) \\ (3,1),(3,2),(3,4),(3,5) \\ (4,1),(4,2),(4,3),(4,5) \\ (5,1),(5,2),(5,3),(5,4) \end{Bmatrix}$$ [1]

X denotes the larger number.

Larger number can be 2,3,4 or 5

When large number is	2	3	4	5
Outcomes	(1, 2), (2, 1)	(1,3), (3, 1), (2, 3), (3, 2)	(1, 4), (4, 1), (4, 2), (2, 4), (3, 4), (4, 3)	(1, 5), (5, 1), (2, 5), (5, 2), (3, 5), (5, 3), (4, 5), (5, 4)
Total	2	4	6	8

[1]

$$P(X = 2) = \frac{2}{20} = \frac{1}{10}$$

$$P(X = 3) = \frac{4}{20} = \frac{2}{10}$$

$$P(X = 4) = \frac{6}{20} = \frac{3}{10}$$

$$P(X = 5) = \frac{8}{20} = \frac{4}{10}$$ [1]

X_i	P_i	P_iX_i	$P_iX_i^2$
2	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{4}{10}$
3	$\frac{2}{10}$	$\frac{6}{10}$	$\frac{18}{10}$
4	$\frac{3}{10}$	$\frac{12}{10}$	$\frac{48}{10}$
5	$\frac{4}{10}$	$\frac{20}{10}$	$\frac{100}{10}$
Total		$\sum P_iX_i = \frac{40}{10} = 4$	$\sum P_iX_i = \frac{170}{10} = 17$

[1]

Mean$=\sum P_iX_i = 4$ [1]

Variance = $\sum P_iX_i^2 - (\text{mean})^2$

$= 17 - (4)^2 = 17 - 16 = 1$ [1]

16. $p = P(red\ ball) = \frac{6}{9} = \frac{2}{3}$

$\therefore q = 1 - p = 1 - \frac{2}{3} = \frac{1}{3}$ [1]

N = 4,

Random variable X can take values 0,1,2,3,4

Using $P(r) = {}^nC_r . q^{n-r} p^r$ [1]

$P(X=0) = {}^4C_0 . q^4 p^0 = q^4 = \left(\frac{1}{3}\right)^4 = \frac{1}{81}$

$P(X=1) = {}^4C_1 . q^3 p^1 = 4\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right) = \frac{8}{81}$

$P(X=2) = {}^4C_2 . q^2 p^2 = 6q^2 p^2 = 6\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 = \frac{24}{81}$

[1]

$P(X=3) = {}^4C_3 . q^1 p^3 = 4q^1 p^3 = 4\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3 = \frac{32}{81}$

$P(X=4) = {}^4C_4 . q^0 p^4 = p^4 = \left(\frac{2}{3}\right)^4 = \frac{16}{81}$

Probability distribution is:

X_i	P_i	P_iX_i	$P_iX_i^2$
0	$\frac{1}{81}$	0	0
1	$\frac{8}{81}$	$\frac{8}{81}$	$\frac{8}{81}$
2	$\frac{24}{81}$	$\frac{48}{81}$	$\frac{96}{81}$
3	$\frac{32}{81}$	$\frac{96}{81}$	$\frac{288}{81}$
4	$\frac{16}{81}$	$\frac{64}{81}$	$\frac{256}{81}$
		$\Sigma P_iX_i = \frac{216}{81}$	$\Sigma P_iX_i^2 = \frac{648}{81}$

[1]

Mean = $\Sigma P_iX_i = \frac{216}{81} = \frac{8}{3}$ [1]

Variance = $\Sigma P_iX_i^2$ - (mean)^2 $= \frac{648}{81} - \left(\frac{8}{3}\right)^2$

$\Rightarrow$ variance $= \frac{72-64}{9} = \frac{8}{9}$ [1]

17. Given: Let E_1, E_1 *and* A be the events defined as follows

E_1 : the student is a hosteler

E_2 : the student does not reside in the hostel

A : student attains A grade [1]

$P(E_1) = 60\% = \frac{60}{100} = \frac{6}{10}$

$P(A \mid E_1) = 30\% = \frac{30}{100} = \frac{3}{10}$ [1]

$P(E_2) = 40\% = \frac{40}{100} = \frac{4}{10}$

$P(A \mid E_2) = 20\% = \frac{20}{100} = \frac{2}{10}$ [1]

Be Bayes' theorem

$P(E_1 \mid A)$

$$= \frac{P(E_1)P(A \mid E_1)}{[P(E_1)P(A \mid E_1)] + [P(E_2)P(A \mid E_2)]} \quad [1]$$

$$= \frac{\left(\frac{6}{10} \times \frac{3}{10}\right)}{\left(\frac{6}{10} \times \frac{3}{10}\right) + \left(\frac{4}{10} \times \frac{2}{10}\right)} \quad [1]$$

$$= \frac{18}{18+8} = \frac{18}{26} = \frac{9}{13} \quad [1]$$

18. Let event $E_1 : A$ male is selected [1]

Let event $E_2 : A$ female is selected

Let event A: selected person is grey haired [1]

$P(E_1) = 50\% = 0.5$

$P(A \mid E_1) = 5\% = 0.05$

$P(E_2) = 50\% = 0.5$ [1]

$P(A \mid E_2) = 0.25\% = 0.0025$

Using Bayes' theorem

$$P(E_1 \mid A) = \frac{P(E_1)\, x P(A \mid E_1)}{(P(E_1) \times P(A \mid E_1)) + (P(E_2) \times P(A \mid E_2))}$$

[1]

$$= \frac{0.5 \times 0.05}{(0.5 \times 0.05) + (0.5 \times 0.0025)} \quad [1]$$

$$= \frac{0.025}{0.025 + 0.00125} = \frac{0.025}{0.02625}$$

$$= \frac{25}{2625} \times 100 = \frac{20}{21} \quad [1]$$

19. If there are 3 coins.

Let these are A, B, C respectively

For coin A $\rightarrow$ Prob. of getting Head P (H) =1

For coin B $\rightarrow$ Prob. of getting Head P (H) = $\frac{3}{4}$ [1]

For coin C $\rightarrow$ Prob. of getting Head P (H) = 0.6

we have to find $P(A \mid H)$

= Prob. of getting H by coin A [1]

So, we can use formula

$$P(A \mid H) = \frac{P\left(\frac{H}{A}\right)P(A)}{P\left(\frac{H}{A}\right)P(A) + P\left(\frac{H}{B}\right)P(B) + P\left(\frac{H}{C}\right)P(C)}$$

[1]

Here $P(A) = P(B) = P(C) = \frac{1}{3}$

(Prob. of choosing any one coin)

$$P\left(\frac{H}{A}\right) = 1, P\left(\frac{H}{B}\right) = \frac{3}{4}, P\left(\frac{H}{C}\right) = 0.6 \quad [1]$$

Put value in formula so

$$\therefore P\left(\frac{A}{H}\right) = \frac{1.\left(\frac{1}{3}\right)}{1.\left(\frac{1}{3}\right) + \frac{3}{4}.\frac{1}{3} + \frac{1}{3}(0.6)} = \frac{1}{1 + 0.75 + 0.6} \quad [1]$$

$$= \frac{100}{235} = \frac{20}{47} \quad [1]$$

20. Given : Let : E_1 : the patient follow a course of meditation and yoga

E_2 : The patient takes a certain drug [1]

A : The patient suffers a heart attack

$$P(E_1) = P(E_2) = \frac{1}{2} \quad [1]$$

$$P\left(\frac{A}{E_1}\right) = \frac{70}{100} \times \frac{40}{100} = \frac{280}{1000} = 0.28 \quad [1]$$

$$P\left(\frac{A}{E_2}\right) = \frac{75}{100} \times \frac{40}{100} = \frac{300}{1000} = 0.30 \quad [1]$$

Required probability

$$P\left(\frac{E_1}{A}\right) = \frac{\left[P(E_1) \times P\left(\frac{A}{E_1}\right)\right]}{\left[P(E_1) \times P\left(\frac{A}{E_1}\right)\right] + \left[P(E_2) \times P\left(\frac{A}{E_2}\right)\right]}$$

[1]

$$= \frac{\frac{1}{2} \times 0.28}{\left(\frac{1}{2} \times 0.28\right) + \left(\frac{1}{2} \times 0.30\right)}$$

$$= \frac{0.14}{0.14 + 0.15} = \frac{0.14}{0.29} = \frac{14}{29} \quad [1]$$

Value Based Questions

PREVIOUS YEARS'

EXAMINATION QUESTIONS

4 Marks Questions

1. Often it is taken that a truthful person commands more respect in the society. A man is known to speak the truth 4 out of 5 times. He throws a die and reports that it is a six. Find the probability that it is actually a six. Do you also agree that the value of truthfulness leads to more respect in the society?

[DELHI 2017]

2. The probabilities of two students A and B coming to the school on time are $\frac{3}{7}$ and $\frac{5}{7}$ respectively. Assuming that the events, 'A coming on time' and 'B coming on time' are independent, find the probability of only one of them coming to the school in time. Write at least one advantage of coming to school on time.

[ALL INDIA 2013]

Solutions

1. Let A be the event that the man reports that six occurs in the throwing of the die.

Let E_1 be the event that six occurs and E_2 be the event that six does not occur. [1]

$\therefore P(E_1) = \frac{1}{6}$ and $P(E_2) = 1 - \frac{1}{6} = \frac{5}{6}$

$P(A \mid E_1) = \frac{4}{5}$, $P(A \mid E_2) = \frac{1}{5}$

$P(E_1 \mid A)$

$$= \frac{P(E_1) \times P(A \mid E_1)}{[P(E_1) \times P(A \mid E_1)] + [P(E_2) \times P(A \mid E_2)]} \quad [1]$$

$$= \frac{\frac{1}{6} \times \frac{4}{5}}{\left[\frac{1}{6} \times \frac{4}{5}\right] + \left[\frac{5}{6} \times \frac{1}{5}\right]}$$

$$= \frac{4}{4+5}$$

$$= \frac{4}{9} \quad [1]$$

Yes, I agree that the value of truthfulness leads to more respect in the society. [1]

2. If P(A coming on time) = $\frac{3}{7}$ [1]

P(B coming on time) = $\frac{5}{7}$ [1]

P(A not coming on time) = $\frac{4}{7}$

P(B not come on school time) = $\frac{2}{7}$

P(only one of them coming on time)

$= P(A) \times P(\bar{B}) + P(\bar{A}).P(B)$

$= \frac{3}{7} \times \frac{2}{7} + \frac{5}{7} \times \frac{4}{7} = \frac{26}{49}$ [1]

Works get completed on time and it is a good habit. [1]

Smart Notes ....

Smart Notes ....

CBSE
Sample Question Paper 1

Mathematics
Class XII

Time : 3 hrs **Maximum Marks : 100**

General Instructions

(i) All questions are compulsory.
(ii) The question paper consists of 29 questions divided into four sections A, B, C and D.
(iii) Section A contains 4 questions of 1 mark each. Section B contains 8 questions of 2 marks each. Section C contains 11 questions of 4 marks each. Section D contains 6 questions of 6 marks each.
(iv) There is no overall choice. However, an internal choice has been provided in four questions of 3 marks each and three questions of 4 marks each. You have to attempt only one of the alternatives in all such questions.
(v) Use of calculators is not permitted.

Section A

(1 × 4 = 4)

Question numbers 1 to 4 carry 1 mark each.

1. Let A be the set of all 100 candidates enrolled for an examination. Let $f: A \to N$ be the function defined by $f(x)$ = registration number of the candidate x. Show that f is one-one but not onto.

2. Evaluate $\Delta = \begin{vmatrix} a & e & d \\ c & a & b \\ a & e & d \end{vmatrix}$ without actually calculating.

3. Find a vector in the direction of vector $a = 3\hat{i} - 4\hat{j}$ having magnitude 5 units.

4. If $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1} x, x > 0$; find the value of x.

SECTION B

(2 × 8 = 16)

Question numbers 5 to 12 carry 2 marks each.

5. Write the function $\tan^{-1}\frac{x}{\sqrt{a^2 - x^2}}, |x| < a$ in the simplest form.

6. Find the equation of a curve passing through $\left(1, \frac{\pi}{3}\right)$ if the slope of the tangent to the curve at any point $A(x, y)$ is $\frac{y}{x} - \sin^2\frac{y}{x}$

7. Find $2A^2 + 7A - 3I$, if $A = \begin{bmatrix} 2 & -5 & 7 \\ -9 & 1 & 0 \\ 4 & 0 & -1 \end{bmatrix}$

8. If, $\tan^{-1}\left(\frac{x-2}{x-3}\right) + \tan^{-1}\left(\frac{x+2}{x+3}\right) = \frac{\pi}{4}$, then find the value of x.

9. Find the equation of all lines having slope 2 and being tangent to the curve $y + \frac{2}{x-3} = 0$.

10. Find the position vector of the mid-point of the vector joining the points $A(4, -1, 5)$ and $B(-2, 0, 3)$.

11. If $f(x) = \begin{cases} \frac{x^3 + x^2 - 16x + 20}{(x-2)^2}, & x \neq 2 \\ k, & x = 2 \end{cases}$ is continuous at $x = 2$, find the value of k.

12. Find the integral of $\frac{2x^2 - 4x + 1}{\sqrt{x}}$.

Section C

(4 × 11 = 44)

Question numbers 13 to 23 carry 4 marks each.

13. Probability of completing a specific task independently by A and B are $\frac{1}{3}$ and $\frac{1}{4}$ respectively. If both of them try to complete the task independently, then find the probability that

i. the task is completed.

ii. exactly one of them completed the task.

14. Find the area of the region bounded by the curve $y^2 = 9x$ and the line $x = 5$.

15. Show that the function $f : R \to R$ given by $f(x) = x^3 + x$ is a bijection.

16. Verify Mean Value Theorem, if $f(x) = 2x^2 - 6x - 4$ in the interval $[a, b]$, where $a = 2$ and $b = 5$. Find all $c \in (2, 5)$ for which $f'(c) = 0$.

OR

Find $\frac{d^2y}{dx^2}$ in terms of y if $y = \sin^{-1}x$.

17. Express the matrix P as the sum of a symmetric and a skew symmetric matrix where

$$P = \begin{bmatrix} -4 & 2 & 1 \\ 3 & -1 & 4 \\ 2 & 0 & 3 \end{bmatrix}$$

OR

Find the inverse of the matrix $A = \begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix}$ if it exists.

18. Find the area enclosed by the curve $x = 3\cos t$, $y = 2\sin t$.

19. If three vectors $a = \hat{i} + 2\hat{j} - 4\hat{k}$, $b = 3\hat{j} + \hat{k}$ and $c = 3\hat{i} - 2\hat{j} + \hat{k}$ are such that $a + \lambda b \perp c$, then find the value of λ.

20. If $x = \sqrt{a^{\sin^{-1}t}}, y = \sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

21. Prove that the lines x = ay + b, z = cy + d and $x = a'y + b', z = c'y + d'$ are perpendicular if aa′ + cc′ + 1 = 0

OR

Find the angle between the lines whose direction cosines are given by the equations $l + m + n = 0,\ l^2 + m^2 - n^2 = 0.$

22. Evaluate $\int \frac{x^2 dx}{x^4 - 5x^2 + 6} dx.$

23. Find two positive numbers such that their sum is 40 and the product of their squares is maximum.

SECTION D

(6 × 6 = 36)

Question numbers 24 to 29 carry 6 marks each.

24. Solve the following LPP graphically:

Maximize $Z = 5x + 7y$

Subject to

$x + y \leq 4$

$3x + 8y \leq 24$

$10x + 7y \leq 35$

$x + y \geq 0$

25. The sum of three numbers is 6. If we multiply the third number by 2 and add the first number to the result, we get 7. By adding second and third numbers to three times the first number, we get 12. Find the numbers using matrix method.

OR

For what value of a and b, the following system of equations is consistent?

$x + y + z = 6$

$2x + 5y + az = b$

$x + 2y + 3z = 14$

26. **(a)** Find the equations of the line passing through the point (2, 0, –1) and parallel to the planes $2x + 4y = 0$ and $6y - 2z = 0$.

(b) Find the equations of the plane that passes through three points:

(2, 4, 1), (–2, 0, 4), (6, 1, –3)

27. A shopkeeper sells three types of flower seeds A_1, A_2 and A_3. They are sold as a mixture where the proportions are 4 : 4 : 2 respectively. The germination rates of the three types of seeds are 45%, 60% and 35%. Calculate the probability

(i) of a randomly chosen seed to germinate.

(ii) that it will not germinate given that the seed is of type A_3.

(iii) that it is of the type A_2 given that a randomly chosen seed does not germinate.

OR

There are 5 cards numbered 1 to 5, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on two cards drawn. Find the mean and variance of X.

28. Solve the differential equation:

$$\left(1+y^2\right)(1+\log\log x)dx + xdy = 0 \text{ when } x = 1, y = 1.$$

OR

Solve the differential equation:

$$x\log x\frac{dy}{dx} + y = \frac{2}{x}\log x$$

29. Evaluate

$$\int_0^{\frac{\pi}{2}}(2\log\sin x - \log\sin 2x)dx$$

CBSE
Sample Question Paper 2

Mathematics
Class XII

Time : 3 hrs | **Maximum Marks : 100**

General Instructions

(i) All questions are compulsory.

(ii) The question paper consists of 29 questions divided into four sections A, B, C and D.

(iii) Section A contains 4 questions of 1 mark each. Section B contains 8 questions of 2 marks each. Section C contains 11 questions of 4 marks each. Section D contains 6 questions of 6 marks each.

(iv) There is no overall choice. However, an internal choice has been provided in four questions of 3 marks each and three questions of 4 marks each. You have to attempt only one of the alternatives in all such questions.

(v) Use of calculators is not permitted.

Section A

(1 × 4 = 4)

Question numbers 1 to 4 carry 1 mark each.

1. If $\begin{bmatrix} 3 & x+4 & y-5 \\ a-3 & -4 & 1 \\ -12 & b-8 & z+5 \end{bmatrix} = \begin{bmatrix} -2 & 5 & 2y+3 \\ 7 & 0 & 11 \\ 3c-3 & -12 & -10 \end{bmatrix}$ Find the values of x, y, z, a, b, c.

2. Evaluate the determinant of $\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$

3. What is the value of $\tan^{-1}\tan^{-1}\left[2\cos\cos\left(2\sin^{-1}\sin^{-1}\frac{1}{2}\right)\right]$

4. Let f:{1, 3, 5, 7} $\rightarrow$ {3, 5, 7, 9} and g:{3, 5, 7, 9} $\rightarrow$ {13, 15, 17} be functions defined as f : {1, 3, 5, 7} $\rightarrow$ {3, 5, 7, 9} and $g(3) = g(5) = 13$ and (7) = g(9) = 15. Find *gof*.

SECTION B

(2 × 8 = 16)

Question numbers 5 to 12 carry 2 marks each.

5. Find the projection of the vector $a = \hat{i} + 5\hat{j} + 3\hat{k}$ on the vector $b = \hat{i} + 3\hat{j} + \hat{k}$

6. If A and B are two events such that $P(A) = 0.4$, $P(B) = 0.5$ and $P(A \cup B) = 0.6$.

Find $P\left(\frac{A}{B}\right)$

7. Evaluate $\int_0^{\frac{\pi}{2}} x \cos\cos x dx$

8. Find the differential equation representing the family of curves y = a cos cos(bx + c), a and c being arbitrary constants.

9. The total revenue received from the profit of x units of product is given by (x) = $2x^2 + 18x + 6$. Find the marginal revenue when = 4, where by marginal revenue we mean the rate of change of total revenue with respect to the profit made at instant.

10. Find the slopes of the tangent and the normal to the curve $2x^2 + y + 3y^2 = 4$ at (2, 1)

11. Let B be the set of all 65 students of Medical batch in a central college. Let $f : B \rightarrow N$ be a function defined by $f(x)$ mobile number of student x. Show that f is one but not onto .

12. A cone of height 10.8 cm and the radius of its base is 2.7 cm. Due to high temperature it melted and took shape of a sphere. Find the radius of the sphere.

SECTION C

(4 × 11 = 44)

Question numbers 13 to 23 carry 4 marks each.

13. Find the ratio in which the point $P(m, 4)$ divides the line segment joining the points $A(2, -5)$ and $B(3, 6)$. Also find the value of m

14. Jessica observed that the angle of elevation of the top of the electricity pole as 30°. She walked 60m towards the foot of the pole along ground level and figured the angle of elevation of top of pole as 60°. Find the height of the pole.

15. Prove that $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{6}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{47}{151}$

16. Let $A=\begin{bmatrix}3 & 1\\ 2 & -1\\ -1 & 4\end{bmatrix}$ and $B=\begin{bmatrix}-2 & 5\\ 0 & -1\\ -3 & 1\end{bmatrix}$. Find, $A^T + 2B^T$

OR

Find the area of the triangle with vertices $A(5, 4)$, $B(-1, 6)$, $C(2, 4)$ and show whether the given points are collinear or not?

17. Evaluate $\int \frac{\cos\theta}{(4+\sin\theta)(2+3\sin\theta)}d\theta$

18. Find the equation of the tangent and the normal to the curve $y = (x^2 - 1)(x + 2)$ at the points where the curve cuts *x-axis*

OR

Find the intervals in which the function $f(x) = 2x^3 - 4x^2 - 8x + 12$ is

(i) Increasing

(ii) Decreasing

19. If the area enclosed between the curves $y = bx^2$ and $x = by^2$ is 2 square unit, then find the value of b

20. If a unit vector p makes angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{6}$ with $\hat{j}$, and an acute angle θ with $\hat{k}$ then find the components of $\hat{p}$ and the angle θ

OR

If the points $A(3, -1, 1)$, $B(4, x, 5)$, $C(2, 10, 4)$ are collinear, find the value of x

21. Solve the differential equation:

$$(1-x^2)\frac{dy}{dx}+2xy=\frac{1}{(1-x^2)}$$

22. A random variable P has the following probability distribution:

x_i	-3	-2	0	1	3	4
p_i	0.2	0.3	0.4	0.1	0.5	0.1

SECTION D

(6 × 6 = 36)

Question numbers 24 to 29 carry 6 marks each.

23. Find the equation of the plane containing lines $\frac{x-2}{3}=\frac{y+1}{2}=\frac{z-1}{4}$ and $\frac{x-3}{2}=\frac{y-k}{1}=\frac{z}{1}$ which intersect each other, Hence, find the value of k

24. Thomas has to attend a meeting. From the past experience, it is known that the probabilities that he will come by walking, bus, car or by other mode of transport are respectively $\frac{4}{10},\frac{2}{5},\frac{1}{5},\frac{1}{12}$. The probability that he will be late are $\frac{1}{3},\frac{1}{4},\frac{1}{6}$, if he comes by walking, bus and car respectively, but if he comes by other mode of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by car?

OR

From a batch of 20 hikers preparing for a hike, 12 always finished the hike on time and reported back on time. Four hikers are selected at random from the batch. Find the probability distribution of the number of selected hikers who always finished on time and reported back. Also, find the mean of the distribution. What values are described in the question?

25. Let $B = N \times N$ and let * be a binary operation on B defined by $(p, q) * (r, s) = (ps + qr, qs)$ for all $(p, q), (r, s) \in N \times N$

Show that

(i) * is commutative on B

(ii) * is associative on B

OR

If $f, g : R \rightarrow R$ are defined respectively by $f(x) = 2x^2 + 4x - l$, $g(x) = (x - 2)$

Find

(i) fog

(ii) gof

(iii) fog

(iv) gog

26. Evaluate the following integral

$\int e^x \cos^2 x dx$

27. Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 2$.

28. Find the equation of a plane passing through the point $X(2, -1, 5)$ and parallel to the plane determined by the points $P(1, -1, 2)$, $Q(3, -2, 5)$ and $R(-3, -5, 4)$. Also, find the distance of this plane from the point P.

29. If $P = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{vmatrix}$, find $P^{(-1)}$. Use it to solve the system of equations.

$x + 2y - 3z = 60$

$2x - y + z = 40$

$3x + y + 2z = 20$

OR

Prove that: $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & c+b \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c)$