Solution

SECTION A

1. As same registration number cannot be assigned to two different candidates, therefore the function f must be one-one. As N is the set of all natural numbers and out of all the natural numbers only 100 natural numbers can be the image of any element of A under f. Hence, f is not onto. [1]

2. Using the property of determinants, "If two rows or columns of a determinant are identical, then the value of the determinant is zero." [½]

Here first and third rows of the determinant are identical.

$\therefore \Delta = 0$ [½]

3. $\vec{a} = \sqrt{3^2 + 4^2} = \sqrt{9+16} = 5$ units

The unit vector in the direction of given vector

$\hat{a} = \frac{1}{a}\vec{a} = \frac{1}{5}(3\hat{i} - 4\hat{j})$ [½]

$\therefore$ The vector in the direction of given vector and having magnitude 5 units

$5\hat{a} = 5 \times \frac{1}{5}(3\hat{i} - 4\hat{j}) = 3\hat{i} - 4\hat{j}$ [½]

4. Given that, $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1} x$

$$\Rightarrow 2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \tan^{-1} x$$

$$\Rightarrow 2\left(\tan^{-1} 1 - \tan^{-1} x\right)$$

$$= \tan^{-1} x \left\{\because \left(\tan^{-1} A - \tan^{-1} B\right) = \tan^{-1}\left(\frac{A-B}{1+AB}\right)\right\}$$

[½]

$$\Rightarrow 2\frac{\pi}{4} - 2\tan^{-1} x = \tan^{-1} x$$

$$\Rightarrow \frac{\pi}{2} = 3\tan^{-1} x$$

$$\Rightarrow \frac{\pi}{6} = \tan^{-1} x$$

$$\Rightarrow \tan^{-1}\frac{1}{\sqrt{3}} = \tan^{-1} x$$

$$\Rightarrow x = \frac{1}{\sqrt{3}}$$ [½]

SECTION B

5. Putting $x = a\sin\varphi$ in the given function such that

$\varphi = \sin^{-1}\frac{x}{a}$

So, $\tan^{-1}\frac{x}{\sqrt{a^2 - x^2}} = \tan^{-1}\left(\frac{a\sin\varphi}{\sqrt{a^2 - a^2\sin^2\varphi}}\right)$

$$= \tan^{-1}\left(\frac{a\sin\varphi}{\sqrt{a^2\left(1-\sin^2\varphi\right)}}\right)$$

$$= \tan^{-1}\left(\frac{a\sin\varphi}{a\sqrt{1-\sin^2\varphi}}\right)$$ [½]

$$= \tan^{-1}\left(\frac{\sin\varphi}{\sqrt{1-\sin^2\varphi}}\right)$$

$$= \tan^{-1}\left(\frac{\sin\varphi}{\cos\varphi}\right)\left(\because \sin^2 A + \cos^2 A = 1\right)$$ [½]

$$= \tan^{-1}(\tan\varphi)\left(\because \tan A = \frac{\sin A}{\cos A}\right)$$ [½]

$$= \varphi$$

$$\Rightarrow \tan^{-1}\frac{x}{\sqrt{a^2 - x^2}} = \varphi = \sin^{-1}\frac{x}{a}$$ [½]

6. Given that, tangent to the curve

$$\frac{dy}{dx} = \frac{y}{x} - \sin^2\frac{y}{x} \quad \text{.....(i)}$$ [½]

The given differential equation is a homogeneous equation, so by substituting y by vx,

$$y = vx$$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

from (i),

$$v + x\frac{dv}{dx} = \frac{vx}{x} - \sin^2\frac{vx}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v - \sin^2 v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\sin^2 v$$

$$\Rightarrow \quad \frac{dv}{\sin^2 v} = -\frac{dx}{x} \text{ or } cosec^2 v dv = -\frac{dx}{x}$$ [½]

Integrating both sides, we get $-\cot v = -\log x + c$ where c is the integration constant.

$$\Rightarrow -cot v + log x = c$$

$$\Rightarrow -\cot\frac{y}{x} + log x = c, \text{ as } y = vx$$

As the curve passes through $\left(1, \frac{\pi}{3}\right)$, putting $x = 1$

and $y = \frac{\pi}{3}$ [½]

$$c = -\cot\frac{\pi}{3} + \log 1$$

$$\Rightarrow \quad c = -\frac{1}{\sqrt{3}} (\because \log 1 = 0)$$

Therefore the required equation of the curve is

$$-\cot\frac{y}{x} + log x = -\frac{1}{\sqrt{3}}$$ [½]

7. Given that, $A = \begin{bmatrix} 2 & -5 & 7 \\ -9 & 1 & 0 \\ 4 & 0 & -1 \end{bmatrix}$

$$\therefore A^2 = \begin{bmatrix} 2 & -5 & 7 \\ -9 & 1 & 0 \\ 4 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & -5 & 7 \\ -9 & 1 & 0 \\ 4 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 77 & -15 & 7 \\ -27 & 46 & -63 \\ 4 & -20 & 29 \end{bmatrix}$$

$$\Rightarrow 2A^2 = \begin{bmatrix} 154 & -30 & 14 \\ -54 & 92 & -126 \\ 8 & -40 & 58 \end{bmatrix}$$ [½]

and $7A = 7\begin{bmatrix} 2 & -5 & 7 \\ -9 & 1 & 0 \\ 4 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 14 & -35 & 49 \\ -63 & 7 & 0 \\ 28 & 0 & -7 \end{bmatrix}$ [½]

$$3I = 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$ [½]

$$\therefore 2A^2 + 7A - 3I = \begin{bmatrix} 154 & -30 & 14 \\ -54 & 92 & -126 \\ 8 & -40 & 58 \end{bmatrix} + \begin{bmatrix} 14 & -35 & 49 \\ -63 & 7 & 0 \\ 28 & 0 & -7 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 165 & -65 & 63 \\ -117 & 96 & -126 \\ 36 & -40 & 48 \end{bmatrix}$$ [½]

8. Given that, $\tan^{-1}\left(\frac{x-2}{x-3}\right) + \tan^{-1}\left(\frac{x+2}{x+3}\right) = \frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x-2}{x-3} + \frac{x+2}{x+3}}{1 - \left(\frac{x-2}{x-3}\right)\left(\frac{x+2}{x+3}\right)}\right] = \frac{\pi}{4}$$

$$\left(\text{as } \tan^{-1} A + \tan^{-1} B = \tan^{-1}\frac{A+B}{1-AB}\right)$$

$$\Rightarrow \tan^{-1}\left[\frac{(x-2)(x+3)+(x+2)(x-3)}{(x-3)(x+3)-(x-2)(x+2)}\right]=\frac{\pi}{4}$$ [½]

$$\Rightarrow \frac{x^2+x-6+x^2-x-6}{x^2-9-x^2+4}=\tan\frac{\pi}{4}$$

$$\Rightarrow \frac{2x^2-12}{-5}=1$$

$2x^2-12=-5 \Rightarrow 2x^2-7=0$ [½]

$2x^2=7 \Rightarrow x^2=\frac{7}{2}$

$\Rightarrow x=\pm\sqrt{\frac{7}{2}}$ [1]

9. Tangent to the curve $y+\frac{2}{x-3}=0$ will be

$$\frac{dy}{dx}-\frac{2}{(x-3)^2}=0$$ [½]

$$\Rightarrow \frac{dy}{dx}=\frac{2}{(x-3)^2}=2 \text{ (given } \frac{dy}{dx}=2\text{)}$$

$\Rightarrow (x-3)^2=1$

$\Rightarrow x-3=\pm1$

Taking $x-3=1$, $x=4 \Rightarrow y=-2$ [½]

By taking $x-3=-1, x=2 \Rightarrow y=2$

So, the equation of the tangent passing through $(4,-2)$: [½]

$y-(-2)=2(x-4)$ or $y-2x+10=0$

And the equation of the tangent passing through $(2,2)$:

$y-2=2(x-2)$ or $y-2x+2=0$ [½]

10. Position vector of point $A(4,-1,5)=4\hat{i}-\hat{j}+5\hat{k}$

Position vector of point $B(-2,0,3)=-2\hat{i}+3\hat{k}$ [½]

The position vector of the mid-point (say P) of the vector joining the points A and B will be [½]

$$\overrightarrow{OP}=\frac{(4\hat{i}-\hat{j}+5\hat{k})+(-2\hat{i}+3\hat{k})}{2}$$

$$\overrightarrow{OP}=\frac{(4-2)\hat{i}-\hat{j}+(5+3)\hat{k}}{2}$$ [½]

$$\overrightarrow{OP}=\frac{2\hat{i}-\hat{j}+8\hat{k}}{2}$$

$$\overrightarrow{OP}=\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$$ [½]

11. It is given that $f(x)=k$ at $x=2$ or $f(2)=k$

$$\lim_{x\to2^-} f(x)=\lim_{x\to2^+} f(x)=\lim_{x\to2}\frac{x^3+x^2-16x+20}{(x-2)^2}$$

$$=\lim_{x\to2}\frac{(x+5)(x-2)^2}{(x-2)^2}$$ [½]

$$=\lim_{x\to2}(x+5)=7$$

$\because f$ is continuous at $x=2$, [½]

$$\lim_{x\to2} f(x)=f(2)$$

Hence, $k=7$ [1]

12. Given, $I=\int\frac{2x^2-4x+1}{\sqrt{x}}dx$

or, $I=\int\left(\frac{2x^2}{\sqrt{x}}-\frac{4x}{\sqrt{x}}+\frac{1}{\sqrt{x}}\right)dx$ [½]

$$\Rightarrow I=\int\left(2x^{3/2}-4x^{1/2}+x^{-1/2}\right)dx$$

$$\Rightarrow I=2\frac{x^{5/2}}{5/2}-4\frac{x^{3/2}}{3/2}+\frac{x^{1/2}}{1/2}+C$$ [½]

C being the integration constant.

$$\Rightarrow I=\frac{4}{5}x^{5/2}-\frac{8}{3}x^{3/2}+2\sqrt{x}+C$$ [1]

SECTION B

13. Probability of task completed by A; $P(A) = \frac{1}{3}$

Probability of task not completed by A;

$P(A') = 1 - \frac{1}{3} = \frac{2}{3}$ [½]

Probability of task completed by B; $P(B) = \frac{1}{4}$

Probability of task not completed by B; $P(B') = 1 - \frac{1}{4} = \frac{3}{4}$ [½]

As the task is completed independently by A and B,

$\therefore P(AB) = P(A) \times P(B) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$ [1]

(i) Probability that task is completed (either by A or B) =P(A ∪ B) $= P(A) + P(B)$

$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$ [1]

(ii) Probability that exactly one of them completed the task = $P(A).P(B') + P(B).P(A')$

$= \frac{1}{3} \times \frac{3}{4} + \frac{1}{4} \times \frac{2}{3}$

$= \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$ [1]

14. The region bounded by the curve $y^2 = 9x$ and the line $x = 5$ is shown in the given figure which is OACO.

Area of OACO = 2(Area of OAB)

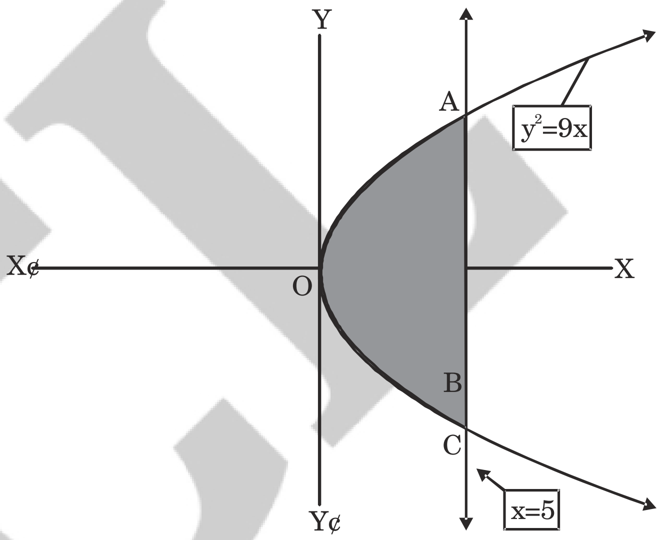
Area of OACO = $2\int_0^5 y\,dx$ [1]

$= 2\int_0^5 3\sqrt{x}\,dx$

$= 6\left[\frac{x^{3/2}}{3/2}\right]_0^5$

$= 4(5)^{3/2}$

$= 20\sqrt{5}\, unit^2$ [1]



[2]

15. To check the injectivity:

Let $x, y \in R$ such that $f(x) = f(y)$

$\Rightarrow x^3 + x = y^3 + y$

$\Rightarrow x^3 - y^3 + x - y = 0$

$\Rightarrow (x - y)(x^2 + xy + y^2 + 1) = 0$

$\Rightarrow x - y = 0$ [$\because x^2 + xy + y^2 \geq 0$ for all $x, y \in R$]

$\Rightarrow x = y$ [1]

Thus, $f(x) = f(y) \Rightarrow x = y$

for all $x, y \in R$

So, f is an injective map.

Now to check surjectivity: Let y be an arbitrary element of R. Then,

$f(x) = y \Rightarrow x^3 + x = y \Rightarrow x^3 + x - y = 0$ [1]

As an odd degree equation has at least one real root. Therefore, for every real value of y, the equation $x^3 + x - y = 0$ has a real root α such that

$\alpha^3 + \alpha - y = 0 \Rightarrow \alpha^3 + \alpha = y \Rightarrow f(\alpha) = y$ [1]

Thus, for every $y \in R$ there exists $\alpha \in R$ such that $f(\alpha) = y$.

So *f* is a surjective map.

Hence, $f: R \to R$ is a bijection. [1]

16. Given, $f(x) = 2x^2 - 6x - 4$

It is a polynomial function, so it is continuous in $[2, 5]$ and also the function is differentiable in $(2, 5)$. [1]

The differential of the given function

$f'(x) = 4x - 6$

$f(a) = f(2) = 2(2)^2 - 6(2) - 4 = -8$

$f(b) = f(5) = 2(5)^2 - 6(5) - 4 = 16$

$\therefore \frac{f(b) - f(a)}{b - a} = \frac{16 - (-8)}{5 - 2} = \frac{24}{3} = 8$

According to Mean Value Theorem, there exists a point $c \in (2, 5)$ such that $f'(c) = 8$ [1]

$f'(c) = 8$

$\Rightarrow 4c - 6 = 8$ [1]

$\Rightarrow c = \frac{14}{4} = \frac{7}{2}$ where $c = \frac{7}{2} \in (2,5)$ [1]

Hence, the Mean Value Theorem is verified.

OR

Given that $y = \sin^{-1} x$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\sin^{-1} x)$ [1]

$= \frac{1}{\sqrt{1 - x^2}} = (1 - x^2)^{-1/2}$

The second order derivative

$\frac{d^2y}{dx^2} = -\frac{1}{2}(1 - x^2)^{-3/2}(-2x)$ [1]

$= \frac{x}{\sqrt{(1 - x^2)^3}}$

$\because y = \sin^{-1} x$

$\Rightarrow x = \sin y$

$\therefore \frac{d^2y}{dx^2} = \frac{\sin \sin y}{\sqrt{(1 - \sin^2 y)^3}}$ [1]

$= \frac{\sin \sin y}{\sqrt{(\cos^2 y)^3}}$

$= \frac{\sin y}{\cos^3 y}$

$= \frac{\sin y}{\cos y} \times \frac{1}{\cos^2 y}$

$\therefore \frac{d^2y}{dx^2} = \tan y . \sec^2 y$ [1]

17. Given matrix, $P = \begin{bmatrix} -4 & 2 & 1 \\ 3 & -1 & 4 \\ 2 & 0 & 3 \end{bmatrix}$

So, $P' = \begin{bmatrix} -4 & 3 & 2 \\ 2 & -1 & 0 \\ 1 & 4 & 3 \end{bmatrix}$ [1]

Let $A = \frac{1}{2}(P + P')$

$= \frac{1}{2}\begin{bmatrix} -8 & 5 & 3 \\ 5 & -2 & 4 \\ 3 & 4 & 6 \end{bmatrix} = \begin{bmatrix} -4 & \frac{5}{2} & \frac{3}{2} \\ \frac{5}{2} & -1 & 2 \\ \frac{3}{2} & 2 & 3 \end{bmatrix}$ [1]

Now $A' = \begin{bmatrix} -4 & \frac{5}{2} & \frac{3}{2} \\ \frac{5}{2} & -1 & 2 \\ \frac{3}{2} & 2 & 3 \end{bmatrix} = A$

Thus $A = \frac{1}{2}(P + P')$ s is a symmetric matrix.

Again, let $B = \frac{1}{2}(P - P')$ [1]

$$= \frac{1}{2}\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 4 \\ 1 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-1}{2} \\ \frac{1}{2} & 0 & 2 \\ \frac{1}{2} & -2 & 0 \end{bmatrix}$$

Now $B' = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & 0 & -2 \\ \frac{-1}{2} & 2 & 0 \end{bmatrix} = -B$

Thus $B = \frac{1}{2}(P - P')$ is a skew-symmetric matrix.

$$\therefore A + B = \begin{bmatrix} -4 & \frac{5}{2} & \frac{3}{2} \\ \frac{5}{2} & -1 & 2 \\ \frac{3}{2} & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-1}{2} \\ \frac{1}{2} & 0 & 2 \\ \frac{1}{2} & -2 & 0 \end{bmatrix}$$

$$\Rightarrow A + B = \begin{bmatrix} -4 & 2 & 1 \\ 3 & -1 & 4 \\ 2 & 0 & 3 \end{bmatrix} = P$$

Hence matrix P is represented as the sum of a symmetric and a skew-symmetric matrix. [1]

OR

Given, $A = \begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix}$

Write, $A = IA$ [1]

i.e. $\begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Applying $R_1 = \frac{1}{2} R_1$ [1]

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $R_1 = R_1 + 2R_2$ [1]

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 2 \\ 0 & 1 \end{bmatrix} A$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{1}{2} & 2 \\ 0 & 1 \end{bmatrix}$$ [1]

18. Given that, $x = 3\cos t, y = 2\sin t$

Eliminating t from both the equations as follows:

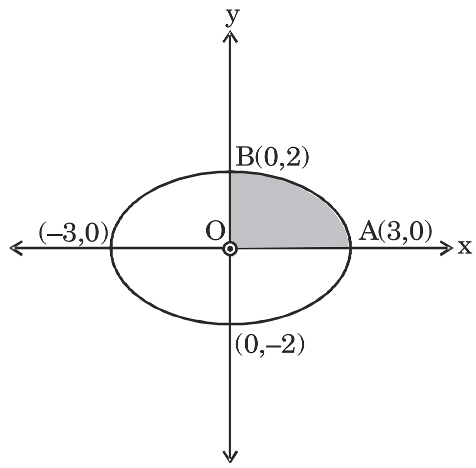
$$\frac{x}{3} = \cos t, \frac{y}{2} = \sin t$$

Squaring and adding both the equations, we get

$$\frac{x^2}{9} + \frac{y^2}{4} = \sin^2 t + \cos^2 t$$

$$\Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$$ [1]

This is the equation of an ellipse.



[2]

From the above figure, we get

$$\text{Area} = 4\int_0^3 \frac{2}{3}\sqrt{9-x^2}\,dx$$

$$= \frac{8}{3}\int_0^3 \sqrt{3^2-x^2}\,dx$$

$$\because \int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

$$\therefore \text{Area} = \frac{8}{3}\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3$$

$\therefore$ Area $= 6\pi$ sq units [1]

19. Given

$\vec{a} = \hat{i} + 2\hat{j} - 4\hat{k}$, $\vec{b} = 3\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} - 2\hat{j} + \hat{k}$

Now, $\vec{a} + \lambda\vec{b} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(3\hat{j} + \hat{k})$ [1]

$$= \hat{i} + (2+3\lambda)\hat{j} + (-4+\lambda)\hat{k}$$

As, $\vec{a} + \lambda\vec{b} \perp \vec{c}$ then $(\vec{a} + \lambda\vec{b})\cdot\vec{c} = 0$ [1]

$$\Rightarrow \left[\hat{i} + (2+3\lambda)\hat{j} + (-4+\lambda)\hat{k}\right]\cdot(3\hat{i} - 2\hat{j} + \hat{k}) = 0$$

$\Rightarrow 3 + (2+3\lambda)(-2) + (-4+\lambda) = 0$ [1]

$$\Rightarrow 3 - 4 - 6\lambda - 4 + \lambda = 0$$

$$\Rightarrow -5 - 5\lambda = 0$$

$$\Rightarrow -5\lambda = 5$$

$\Rightarrow \lambda = -1$ [1]

20. Given, $x = \sqrt{a^{\sin^{-1} t}}, y = \sqrt{a^{\cos^{-1} t}}$ [½]

$$\Rightarrow x = \left(a^{\sin^{-1} t}\right)^{\frac{1}{2}} = a^{\frac{1}{2}\sin^{-1} t}$$

and $y = \left(a^{\cos^{-1} t}\right)^{\frac{1}{2}} = a^{\frac{1}{2}\cos^{-1} t}$ [½]

Taking, $x = a^{\frac{1}{2}\sin^{-1} t}$ and taking log both sides, we get [½]

$$\log x = \frac{1}{2}\sin^{-1} t \log a$$

$\therefore \frac{1}{x}\frac{dx}{dt} = \frac{1}{2}\log a \cdot \frac{d}{dt}(\sin^{-1} t)$ [½]

$$\Rightarrow \frac{dx}{dt} = \frac{x}{2}\log a \cdot \frac{1}{\sqrt{1-t^2}}$$

$\Rightarrow \frac{dx}{dt} = \frac{x\log a}{2\sqrt{1-t^2}}$ [½]

Again taking, $y = a^{\frac{1}{2}\cos^{-1} t}$

Taking log both sides, we get [½]

$$\log y = \frac{1}{2}\cos^{-1} t \log a$$

$$\therefore \frac{1}{y}\cdot\frac{dy}{dt} = \frac{1}{2}\log a \cdot \frac{d}{dt}(\cos^{-1} t)$$

$\Rightarrow \frac{dy}{dt} = \frac{y\log a}{2}\cdot\left(-\frac{1}{\sqrt{1-t^2}}\right)$ [½]

$$\Rightarrow \frac{dy}{dt} = -\frac{y\log a}{2\sqrt{1-t^2}}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{y\log a}{2\sqrt{1-t^2}}}{\frac{x\log a}{2\sqrt{1-t^2}}}$$

Hence, $\frac{dy}{dx} = -\frac{y}{x}$ [½]

21. From line, $x = ay + b, z = cy + d$

$$y = \frac{x-b}{a} \text{ and } y = \frac{z-d}{c}$$

$\Rightarrow \frac{x-b}{a} = \frac{y}{1} = \frac{z-d}{c}$ ....(1) [1]

Similarly from line, $x = a'y + b', z = c'y + d'$

$$y = \frac{x-b'}{a'} \text{ and } y = \frac{z-d'}{c'}$$

$\Rightarrow \frac{x-b'}{a'} = \frac{y}{1} = \frac{z-d'}{c'}$ ....(2) [1]

It can be seen that line (1) is parallel to the vector $a\hat{i}+\hat{j}+c\hat{k}$ [1]

And line (2) is parallel to the vector $\hat{i}+\hat{j}+c'\hat{k}$ [½]

If the lines are perpendicular then [½]

$(a\hat{i}+\hat{j}+c\hat{k})\cdot(a'\hat{i}+\hat{j}+c'\hat{k})=0$

$\Rightarrow aa'+1+cc'=0$ or $aa'+cc'+1=0$

OR

We have, $l+m+n=0,\ l^2+m^2-n^2=0$

Eliminating n from both the equations, we have

$l^2+m^2-(l+m)^2=0$

$\Rightarrow l^2+m^2-l^2-m^2-2lm=0$ [½]

$\Rightarrow 2lm=0$

$\Rightarrow lm=0$

$\Rightarrow l=0$ or $m=0$ [½]

If $l=0$ we have $m+n=0$ and $m^2-n^2=0$

$\Rightarrow l=0, m=\alpha, n=-\alpha$

If $m=0$, we have $l+n=0$ and $l^2-n^2=0$

$\Rightarrow l=-\alpha, m=0, n=\alpha$ [½]

So, the vector parallel to these lines are $\vec{a}=\hat{j}-\hat{k}$ and $\vec{b}=-\hat{i}+\hat{k}$

If angle between the lines is φ, then

$$\cos\varphi = \frac{|\vec{a}\cdot\vec{b}|}{|\vec{a}||\vec{b}|} = \frac{1}{\sqrt{2}\cdot\sqrt{2}}$$ [½]

$$\Rightarrow \cos\varphi = \frac{1}{2}$$

$$\therefore \varphi = \frac{\pi}{3}$$ [1]

22. Given, $I=\int\frac{x^2dx}{x^4-5x^2+6}dx$

Let, $x^2=t$

then $\frac{x^2}{x^4-5x^2+6}=\frac{t}{t^2-5t+6}=\frac{t}{(t-2)(t-3)}$ [½]

Say, $\frac{t}{(t-2)(t-3)}=\frac{A}{t-2}+\frac{B}{t-3}$

$\therefore t=A(t-3)+B(t-2)$ [½]

Comparing coefficients both sides, we get

$A=-2, B=3$ [½]

Now $\frac{x^2}{x^4-5x^2+6}=-\frac{2}{x^2-2}+\frac{3}{x^2-3}$

Then, $I=\int\frac{x^2}{x^4-5x^2+6}dx$

$$=-2\int\frac{1}{x^2-2}dx+3\int\frac{1}{x^2-3}dx$$

or $I=-2\int\frac{1}{x^2-(\sqrt{2})^2}dx+3\int\frac{1}{x^2-(\sqrt{3})^2}dx$ [½]

$$=-\frac{2}{2\sqrt{2}}\log\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right|+\frac{3}{2\sqrt{3}}\log\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right|+C$$

$\left(\because \int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\frac{x-a}{x+a}+C\right)$ [1]

Hence,

$$I=-\frac{\sqrt{2}}{2}\log\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right|+\frac{\sqrt{3}}{2}\log\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right|+C$$ [1]

23. Let one number be x and then the other is $y=40-x$ as their sum is 40.

Let $P(x)=x^2y^2$ or $P(x)=x^2(40-x)^2$ [½]

Now finding the first order derivative;

$P'(x) = 2x(40-x)^2 - 2x^2(40-x)$ [½]

$\therefore P'(x) = 2x(40-x)(40-2x)$

or $P'(x) = (40-x)(80x-4x^2)$

And $P''(x) = -(80x-4x^2)+(40-x)(80-8x)$ [½]

$P''(x) = 12x^2 - 480x + 3200 \Rightarrow 3x^2 - 120x + 800$

Now, $P'(x) = 0$ [½]

$\Rightarrow 2x(40-x)(40-2x) = 0$

$\Rightarrow x = 0, x = 40, x = 20$ [½]

When $x = 0, y = 40 \Rightarrow P(x) = 0$

When $x = 40, y = 0 \Rightarrow P(x) = 0$ [½]

$\therefore x = 0, x = 40$ cannot be the possible values of x.

When $x = 20, y = 20$ [½]

and

$P''(20) = 3(20)^2 - 120(20) + 800 = -400 < 0$ [½]

So, by second derivative test $P(x)$, will be maximum when $x = 20, y = 20$

Hence the require numbers are 20 and 20. [½]

SECTION D

24. Converting the inequities into equations, we get

$x + y = 4, 3x + 8y = 24, 10x + 7y = 35, x = 0, y = 0$

[½]

These equations represent straight lines in the following graph.

The feasible region of the given LPP is shaded. The coordinates of the corner points of the feasible region are: [½]

$O(0,0), A(0,3), B\left(\frac{8}{5},\frac{12}{5}\right), C\left(\frac{7}{3},\frac{5}{3}\right), D\left(\frac{7}{2},0\right)$

Corresponding values of Z are:

$(0,0) \Rightarrow Z = 0$ [½]

$(0,3) \Rightarrow Z = 21$ [½]

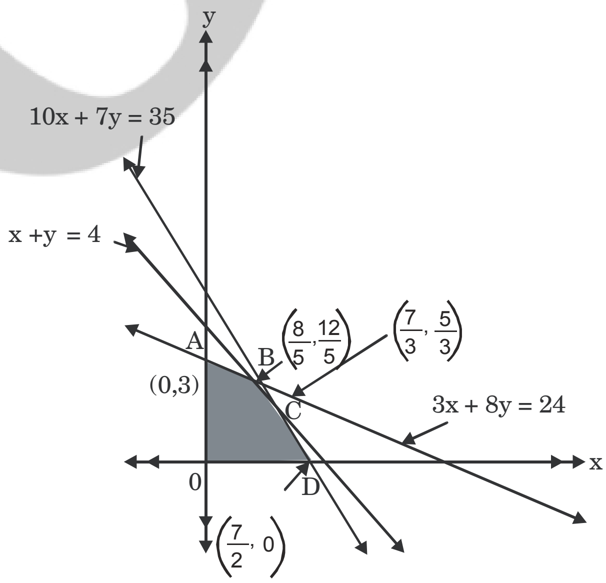
$\left(\frac{8}{5},\frac{12}{5}\right) \Rightarrow Z = 24.8$

$\left(\frac{7}{3},\frac{5}{3}\right) \Rightarrow Z = 23.3$ [½]

$\left(\frac{7}{2},0\right) \Rightarrow Z = 17.5$ [½]

So the maximum value of Z is given by $B\left(\frac{8}{5},\frac{12}{5}\right)$ which is $Z = 24.8$. [1]

[2]



25. Let the three numbers be x, y, z respectively. Then,

$x + y + z = 6$ [½]

$x + 0y + 2z = 7$ [½]

$3x + y + z = 12$ [½]

The above system of equations can be written in matrix form as:

$AX = B$

or $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$ [½]

Now $|A| = 1(-2) - 1(1-6) + 1(1)$

$= -2 + 5 + 1 = 4 \neq 0$ [½]

So, the above system of equations has a uniques solution given by $X = A^{-1}B$. [½]

Now

$A_{11} = -2, A_{12} = 5, A_{13} = 1, A_{21} = 0, A_{22} = -2,$

$A_{23} = 2, A_{31} = 2, A_{32} = -1, A_{33} = -1$ [½]

$\therefore adjA = \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$ [½]

$\therefore A^{-1} = \frac{1}{|A|} adjA = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$ [½]

Now $X = A^{-1}B$

$\Rightarrow X = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$ [½]

$\Rightarrow X = \frac{1}{4}\begin{bmatrix} -12+0+24 \\ 30-14-12 \\ 6+14-12 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \Rightarrow x = 3, y = 1, z = 2$ [½]

Hence the numbers are 3,1 and 2 respectively. [½]

OR

The given system of equations is consistent, if $D \neq 0$ or, if $D = 0$,

then $D_1 = D_2 = D_3 = 0$

We have

$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & a \\ 1 & 2 & 3 \end{vmatrix} = 15 - 2a - 6 + a + 4 - 5 = 8 - a$ [½]

$D_1 = \begin{vmatrix} 6 & 1 & 1 \\ b & 5 & a \\ 14 & 2 & 3 \end{vmatrix} = 6(15-2a) - (3b - 14a) + (2b - 70)$

$= 2a - b + 20$ [½]

$D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 2 & b & a \\ 1 & 14 & 3 \end{vmatrix} = (3b - 14a) - 6(6-a) + (28-b)$

$= -8a + 2b - 8$ [½]

$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 5 & b \\ 1 & 2 & 14 \end{vmatrix} = (70 - 2b) - (28 - b) + 6(4-5)$

$= 36 - b$ [½]

Now, $D \neq 0 \Rightarrow a - 8 \neq 0 \Rightarrow a \neq 8$ [1]

Thus, the given system of equations will be consistent and will have uniques solution for $a \neq 8$.

For $a = 8$, we have [½]

$D = 0$ and $D_1 = 36 - b, D_2 = 2b - 72, D_3 = 36 - b$

Clearly, $D_1 = D_2 = D_3 = 0$ for $b = 36$

Thus, for $a = 8$ and $b = 36$

we have $D = D_1 = D_2 = D_3 = 0$ [½]

Putting $a = 8$ and $b = 36$, the given system of equations reduces to

$x + y + z = 6$

$2x + 5y + 8z = 36$

$x + 2y + 3z = 14$

Taking $z = k$, first and third equations become

$x + y = 6 - k$

$x + 2y = 14 - 3k$ [½]

Solving these equations by Cramer's rule, we get

$$x = \frac{\begin{vmatrix} 6-k & 1 \\ 14-3k & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 12 - 2k - 14 + 3k = k - 2$$

$$y = \frac{\begin{vmatrix} 1 & 6-k \\ 2 & 14-3k \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 14 - 3k - 6 + k = 8 - 2k$$ [½]

Thus, we have $x = k - 2, y = 8 - 2k, z = k$

Clearly, these values satisfy the second equation.

Thus the given system of equations will be consistent and will have infinitely many solutions for $a = 8, b = 36$.

Hence, the given system of equations will be consistent if

$a \neq 8, b \in R$ or if $a = 8, b = 36$ [1]

26. (a) Given, equations of the planes:

$2x + 4y = 0$ and $6y - 2z = 0$

Normal to the plane (1), $\overrightarrow{n_1}\ 2\hat{i} + 4\hat{j}$ [½]

Normal to the plane (2), $\overrightarrow{n_2} = 6\hat{j} - 2\hat{k}$ [½]

The required line is parallel to the given planes, therefore the line is perpendicular to $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$.

The line is parallel to the vector

$\vec{b} = \overrightarrow{n_1} \times \overrightarrow{n_2}$ [½]

$$\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & 0 \\ 0 & 6 & -2 \end{vmatrix}$$

$\vec{b} = \hat{i}(-8-0) - \hat{j}(-4-0) + \hat{k}(12-0)$

$\vec{b} = -8\hat{i} + 4\hat{j} + 12\hat{k}$ [½]

So the equation of the line passing through the point $(2, 0, -1)$ and parallel to the given planes is [½]

$$\frac{x-2}{-8} = \frac{y}{4} = \frac{z+1}{12}$$ [½]

(b) Given points are:

$A = (2, 4, 1), B = (-2, 0, 4), C = (6, 1, -3)$

$$\begin{vmatrix} 2 & 4 & 1 \\ -2 & 0 & 4 \\ 6 & 1 & -3 \end{vmatrix} = 2(-4) - 4(6 - 24) + 1(-2)$$ [1]

$= -8 + 72 - 2 = 62 \neq 0$

Therefore, a plane will pass through the points A,B and C.

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$ [½]

$$\Rightarrow \begin{vmatrix} x-2 & y-4 & z-1 \\ -4 & -4 & 3 \\ 4 & -3 & -4 \end{vmatrix} = 0$$ [½]

$\Rightarrow (x-2)(16+9) - (y-4)(16-12)$

$+(z-1)(12+16) = 0$

$\Rightarrow 25(x-2) - 4(y-4) + 28(z-1) = 0$

$\Rightarrow 25x - 50 - 4y + 16 + 28z - 28 = 0$

$\Rightarrow 25x - 4y + 28z = 62$ [1]

This is the required equation of the plane.

27. Given that,

$$A_1 : A_2 : A_3 = 4:4:2$$

$$P(A_1) = \frac{4}{10}, P(A_2) = \frac{4}{10} \text{ and } P(A_3) = \frac{2}{10} \quad [1]$$

Let E be the event that a seed germinates and E' be the event that a seed does not germinate.

$$\therefore P\left(\frac{E}{A_1}\right) = \frac{45}{100}, P\left(\frac{E}{A_2}\right) = \frac{60}{100}, P\left(\frac{E}{A_3}\right) = \frac{35}{100}$$

[½]

$$P\left(\frac{E'}{A_1}\right) = \frac{55}{100}, P\left(\frac{E'}{A_2}\right) = \frac{40}{100}, P\left(\frac{E'}{A_3}\right) = \frac{65}{100}$$

[½]

(i) $\therefore P(E) = P(A_1)\cdot P\left(\frac{E}{A_1}\right) + P(A_2)\cdot P\left(\frac{E}{A_2}\right) + P(A_3)\cdot P\left(\frac{E}{A_3}\right)$ [½]

$$= \frac{4}{10}\cdot\frac{45}{100} + \frac{4}{10}\cdot\frac{60}{100} + \frac{2}{10}\cdot\frac{35}{100} \quad [½]$$

$$= \frac{180}{1000} + \frac{240}{1000} + \frac{70}{1000} = 0.49$$

(ii) $P\left(\frac{E'}{A_3}\right) = 1 - P\left(\frac{E}{A_3}\right) = 1 - \frac{35}{100} = \frac{65}{100}$ [1]

(iii) $P\left(\frac{A_2}{E'}\right) =$

$$\frac{P(A_2)\cdot P\left(\frac{E'}{A_2}\right)}{P(A_1)\cdot P\left(\frac{E'}{A_1}\right) + P(A_2)\cdot P\left(\frac{E'}{A_2}\right) + P(A_3)\cdot P\left(\frac{E'}{A_3}\right)}$$

[½+½]

$$= \frac{\frac{4}{10}\cdot\frac{40}{100}}{\frac{4}{10}\cdot\frac{55}{100} + \frac{4}{10}\cdot\frac{40}{100} + \frac{2}{10}\cdot\frac{65}{100}} = \frac{16}{51} \quad [½]$$

OR

Here, $S =$

$$\left\{\begin{matrix}(1,2),(2,1),(1,3),(3,1),(1,4),(4,1),(1,5),(5,1),(2,3),(3,2),(2,4), \\ (4,2),(2,5),(5,2),(3,4),(4,3),(3,5),(5,3),(4,5),(5,4)\end{matrix}\right\}$$

$\Rightarrow n(S) = 20$ [½]

Let random variable be X which denotes the sum of the numbers on the cards drawn.

$\therefore X = 3,4,5,6,7,8,9$

At $X = 3, P(X) = \frac{2}{20} = \frac{1}{10}$ [½]

At $X = 4, P(X) = \frac{2}{20} = \frac{1}{10}$ [½]

At $X = 5, P(X) = \frac{4}{20} = \frac{1}{5}$ [½]

At $X = 6, P(X) = \frac{4}{20} = \frac{1}{5}$ [½]

At $X = 7, P(X) = \frac{4}{20} = \frac{1}{5}$ [½]

At $X = 8, P(X) = \frac{2}{20} = \frac{1}{10}$ [½]

At $X = 9, P(X) = \frac{2}{20} = \frac{1}{10}$ [½]

Now, mean $E(X) = \sum XP(X)$

$$= \frac{3}{10} + \frac{4}{10} + \frac{5}{5} + \frac{6}{5} + \frac{7}{5} + \frac{8}{10} + \frac{9}{10} \quad [½]$$

$$= \frac{3+4+10+12+14+8+9}{10} = 6$$

Also,

$$\sum X^2 P(X) = \frac{9}{10} + \frac{16}{10} + \frac{25}{5} + \frac{36}{5} + \frac{49}{5} + \frac{64}{10} + \frac{81}{10}$$

$$= \frac{9+16+50+72+98+64+81}{10} = 39 \quad [½]$$

And, $Var(X) = \sum X^2 P(X) - \left[\sum X(PX)\right]^2$

$$= 39 - 6^2 = 39 - 36 = 3 \quad [1]$$

28. The given differential equation is:

$$\left(1+y^2\right)(1+\log x)\,dx+xdy=0$$

$$\Rightarrow (1+\log x)\left(1+y^2\right)dx=-xdy$$

$$\Rightarrow \frac{1+logx}{x}dx=-\frac{1}{1+y^2}dy \quad [½]$$

$$\Rightarrow \int\frac{1+logx}{x}dx=-\int\frac{1}{1+y^2}dy \quad [½]$$

Let $1+\log x=t\Rightarrow\frac{1}{x}dx=dt$

$$\Rightarrow \int tdt=-\int\frac{1}{1+y^2}dy \quad [½]$$

$$\Rightarrow \frac{t^2}{2}=-\tan^{-1}y+C \quad [½]$$

$$\Rightarrow \frac{1}{2}(1+logx)^2=-\tan^{-1}y+C\ldots(i) \quad [½]$$

Now putting $x=1, y=1$ in (i),we get

$$\frac{1}{2}(1+\log 1)^2=-\tan^{-1}1+C \quad [½]$$

$$\frac{1}{2}=-\frac{\pi}{4}+C\Rightarrow C=\frac{1}{2}+\frac{\pi}{4} \quad [1]$$

Putting $C=\frac{1}{2}+\frac{\pi}{4}$ in (i), we get

$$\frac{1}{2}(1+logx)^2=-\tan^{-1}y+\frac{1}{2}+\frac{\pi}{4} \quad [½]$$

$$\Rightarrow \tan^{-1}y=\frac{\pi}{4}+\frac{1}{2}-\frac{1}{2}(1+logx)^2 \quad [½]$$

$$\Rightarrow y=\tan\left\{\frac{\pi}{4}+\frac{1}{2}-\frac{1}{2}(1+logx)^2\right\} \quad [1]$$

OR

The given differential equation is:

$$x\log x\frac{dy}{dx}+y=\frac{2}{x}\log x$$

$$\Rightarrow \frac{dy}{dx}+\frac{1}{xlogx}y=\frac{2}{x^2}\ldots(i) \quad [½]$$

This is a linear differential equation of the form

$$\frac{dy}{dx}+Py=Q,$$

where $P=\frac{1}{xlogx}, Q=\frac{2}{x^2}$ [½]

$$\therefore I.F.=e^{\int Pdx}=e^{\int\frac{1}{xlogx}dx}$$

Let $\log x=t$

Then $I.F.=e^{\int\frac{1}{t}dt}=e^{logt}=t=\log x$ [½]

Multiplying both sides of (i) by I.F.,we get

$$\log x\frac{dy}{dx}+\frac{1}{x}y=\frac{2}{x^2}\log x \quad [1]$$

Integrating both sides with respect to x, we get

$$y\log x=\int\frac{2}{x^2}\log xdx+C \quad [½]$$

$$\Rightarrow y\log x=2\int\log x\,x^{-2}dx+C \quad [½]$$

$$\Rightarrow y\log x=2\{\log x\left(\frac{x^{-1}}{-1}\right)-\int\frac{1}{x}\left(\frac{x^{-1}}{-1}\right)dx\}+C \quad [½]$$

$$\Rightarrow y\log x=2\{-\frac{logx}{x}+\int x^{-2}dx\}+C \quad [½]$$

$$\Rightarrow y\log x=2\left\{-\frac{logx}{x}-\frac{1}{x}\right\}+C \quad [½]$$

$$\Rightarrow y\log x=-\frac{2}{x}(1+\log x)+C \quad [1]$$

29. Given $I=\int_0^{\frac{\pi}{2}}(2\log sinx-\log\sin 2x)dx$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\{2\log sinx-\log\left(2sinxcox\right)\}dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\left(2logsinx-logsinx-logcosx-\log 2\right)dx$$

[½]

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\left(logsinx-logcosx-\log 2\right)dx\ldots(i)$$ [½]

It is known that,

$$\left(\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right)$$ [1]

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\left[\log\left(\frac{\pi}{2}-sinx\right)-\log\left(\frac{\pi}{2}-cosx\right)-\log 2\right]dx$$

[½]

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\left(logcosx-logsinx-\log 2\right)dx\ldots(ii)$$

Adding (i) and (ii), we get [½]

$$2I=\int_0^{\frac{\pi}{2}}\left(-\log 2-\log 2\right)dx$$ [1]

$$\Rightarrow 2I=-2\log 2\int_0^{\frac{\pi}{2}}dx$$ [½]

$$\Rightarrow I=-\log 2\left[x\right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow I=-\log 2\left[\frac{\pi}{2}\right]$$ [½]

$$\Rightarrow I=\frac{\pi}{2}(-\log 2)$$

$$\Rightarrow I=\frac{\pi}{2}\left[\log\frac{1}{2}\right]$$

$$\Rightarrow I=\frac{\pi}{2}\log\frac{1}{2}$$ [1]