[TOPIC 1] Concept of Relations and Functions

Summary

Relation

Definition: If $(a, b) \in R$, we say that a is related to b under the relation R and we write it as aRb.

Domain of a relation: The set of first components of all the ordered pairs which belong to R is the domain of R.

Domain $(R) = \{a \in A : (a, b) \in R \,\forall\, b \in B\}$

Range of a relation: The set of second components of all the ordered pairs which belong to R is the domain of R.

Range of $R = \{b \in B : (a, b) \in R \,\forall\, a \in A\}$

Types of relations:

- **Empty relation:** Empty relation is the relation R from X to Y if no element of X is related to any element of Y, it is given by $R = \varphi \subset X \times Y$.

  For example, let $X = \{2,4,6\}$, $Y = \{8,10,12\}$

  $R = \{(a, b) : a \in X, b \in Y \text{ and } a + b \text{ is odd}\}$

  R is an empty relation.
- **Universal relation:** Universal relation is a relation R from X to Y if each element of X is related to every element of Y it is given by $R = X \times Y$.

  For example, let $X = \{x, y\}$, $Y = \{x, z\}$

  $R = \{(x, x), (y, z), (y, x), (y, z)\}$

  $R = X \times Y$, so relation R is a universal relation.
- **Reflexive relation:** Reflexive relation R in X is a relation with $(a, a) \in R \,\forall a \in X$.

  For example, let $X = \{x, y, z\}$ and relation R is given as

  $R = \{(x, x), (y, y), (z, z)\}$

  Here, R is a reflexive relation on X.
- **Symmetric relation:** Symmetric relation R in X is a relation satisfying $(a, b) \in R$ implies $(b, a) \in R$.

  For example, let $X = \{x, y\ z\}$ and relation R is given as

  $$R = \{(x, y), (y, x)\}$$

  Here, R is a symmetric relation on X.
- **Transitive relation:** Transitive relation R in X is a relation satisfying $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, b) \in R$.

  For example, let $X = \{x, y, z\}$ and relation R is given as

  $$R = \{(x, z), (z, y), (x, y)\}$$

  Here, R is a transitive relation on X.
- **Equivalence relation:** It is a relation R in X which is reflexive, symmetric and transitive.

  For example, let $X = \{x, y, z\}$ and relation R is given as

  $R = \{(x, y), (x, x), (y, x), (y, y), (z, z), (x, z), (z, x), (y, z)\}$

  Here, R is reflexive, symmetric and transitive. So R is an equivalence relation on X.
- Equivalence class $[a]$ containing $a \in X$ for an equivalence relation R in X is the subset of X containing all elements b related to a.

Function

Definition: A rule f which associates each element of a non-empty set A with a unique element of another non-empty set B is called a function.

Types of functions:

- **Injective function:** A function $f : X \to Y$ is one-one (or injective) if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \forall x_1, x_2 \in X$.
- **Surjective function:** A function $f : X \to Y$ is onto (or surjective) if given any $y \in Y$, $\exists x \in X$ such that $f(x) = y$.
- **Bijective function:** A function $f : X \to Y$ is one-one and onto (or bijective), if f is both one-one and onto.
- **Composite function:** The composition of functions $f : A \to B$ and $g : B \to C$ is the function $gof : A \to C$ given by $gof(x) = g(f(x)) \forall\, x \in A$
- **Invertible function:** A function $f : X \to Y$ is invertible if $\exists g : Y \to X$ such that $gof = I_X$ and $fog = I_Y$.

  A function $f : X \to Y$ is invertible if and only if f is one-one and onto.
- **Steps to find inverse of a function**

  Let $f(x) = y$ where $x \in X$ and $y \in Y$

  Solve $f(x) = y$ for x in terms of y.

  Now replace x with $f^{-1}(y)$ in the expression obtain from the above step.

  Finally to find the inverse function of $f f^{-1}(x)$ replace y with x in the expression obtained from the above step.

[TOPIC 2] Binary Operations

Summary

- **Binary Operation:** A binary operation * on a set A is a function * from $A \times A$ to A. We denote * (a, b) by $a * b$.
- An element $e \in X$ is the **identity element** for binary operation * : $X \times X \rightarrow X$, if $a * e = a = e * a \forall a \in X$.
- An element $a \in X$ is **invertible** for binary operation * : $X \times X \rightarrow X$, if there exists $b \in X$ such that $a * b = e = b *$ a where, e is the identity for the binary operation *. The element b is called inverse of a and is denoted by a^{-1}.
- An operation * on X is **commutative** if $a * b = b * a \forall a, b$ in X.
- An operation * on X is **associative** if
  $(a * b) * c = a * (b * c) \forall a, b, c$ in X.

PREVIOUS YEARS'

EXAMINATION QUESTIONS
TOPIC 2

1 Mark Questions

1. Let * be a binary operation, on the set of all non zero real numbers, given by $a * b = \frac{ab}{5}$ for all $a, b \in R - \{0\}$. Find the value of x, given that $2 * (x * 5) = 10$.

   [DELHI 2014]

2. Let * be a 'binary operation on N given $a * b$ = LCM (a, b) for all $a, b \in N$. find 5*7 .

   [ALL INDIA 2012]

3. Let * be a binary operation on N given by a * b = LCM (a, b) for all a,b $\in$ N. Find 5*7.

   [DELHI 2012]

4. If a*b denote the larger of 'a' and 'b' and if $a \bigcirc b = (a * b) + 3$, then write the value of $5 \bigcirc 10$, where * and O are binary operations.

   [DELHI 2018]

5. Consider the binary operation *on the set {1, 2, 3, 4, 5} defined by a*b = min {a, b} write the operation table of the operation .

   [DELHI 2011]

6 Marks Questions

6. Let A = Q × Q, where Q is the set of all rational numbers, and be a binary operation defined on A by $(a, b) * (c, d) = (ac, b + ad)$, for all (a, b) (c, d) A. Find

   (i) the identity element in A.

   (ii) the invertible element of A.

   [ALL INDIA 2015]

7. Discuss the Commutativity and associativity of binary operation * defined on $A = Q - \{1\}$ by the rule $a * b = a - b + ab \ \forall a, b \in A$. Also find the identity element of * in A and hence find the invertible elements of A.

   [DELHI 2017]

Solutions

1. Given that, $2 * (x * 5) = 10$

   $$2 * \left(\frac{x5}{5}\right) = 10 \left(\because a * b = \frac{ab}{5}\right)$$ [½]

   $$\Rightarrow 2 * x = 10$$

   $$\Rightarrow \frac{2x}{5} = 10$$

   $$\therefore x = 10 \times \frac{5}{2} = 25$$ [½]

2. Given a*b = LCM (a, b)

   5 * 7 = LCM (5, 7) = 35 [1]

Inverse Trigonometric Functions

Summary

Definition of inverse trigonometric functions:

Inverse trigonometric functions are the inverse of trigonometric functions, we can represents them by using “-1” or arc on trigonometric functions. Also the Range of trigonometric function becomes the Domain of Inverse trigonometric function.

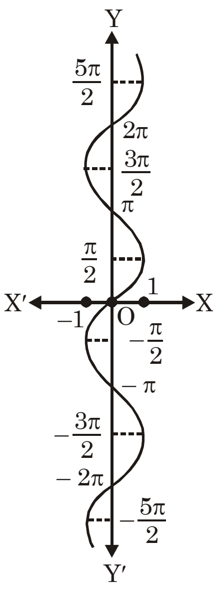
For ex: $x = \sin y$ will be represented as $y = \arcsin x$ or $y = \sin^{-1} x$.

Range of $x = \sin y$ is [–1, 1] and Domain of $y = \arcsin x$ is [–1, 1].

- The inverse trigonometric functions are also called as **Inverse Circular Functions**.
- **Function:** $y = \sin^{-1}x$

  Domain: [–1, 1]

  Range: $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

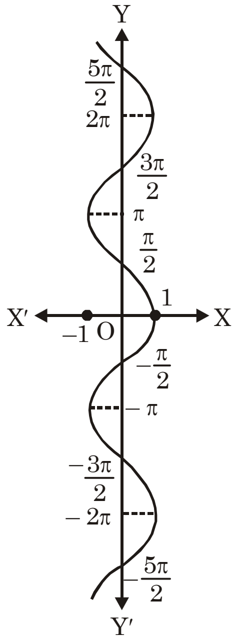


$y = \sin^{-1} x$

- **Function:** $y = \cos^{-1}x$

  Domain: [–1, 1]

  Range: [0, π]

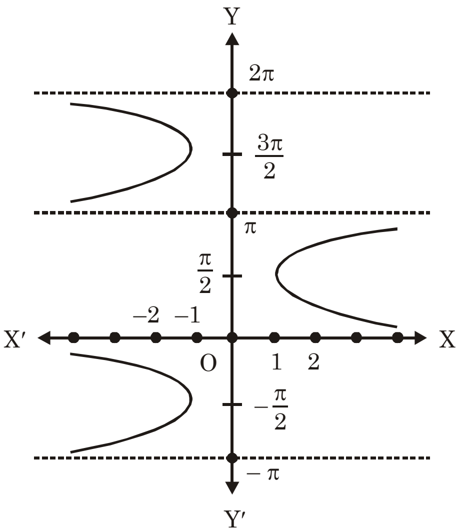


$y = \cos^{-1} x$

- **Function:** $y = \text{cosec}^{-1}x$

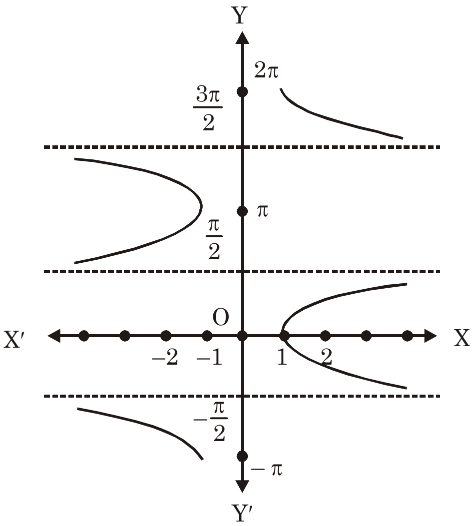
  Domain: R – (–1, 1)

  Range: $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$



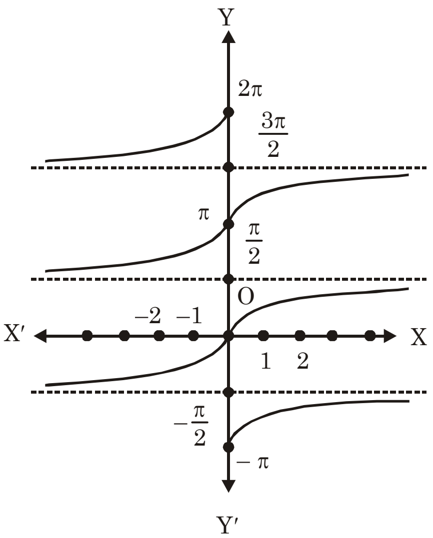
$y = \text{cosec}^{-1} x$

- **Function:** $y = \sec^{-1}x$
  Domain: R – (–1, 1)
  Range: $[0, \pi] - \left\{\frac{\pi}{2}\right\}$



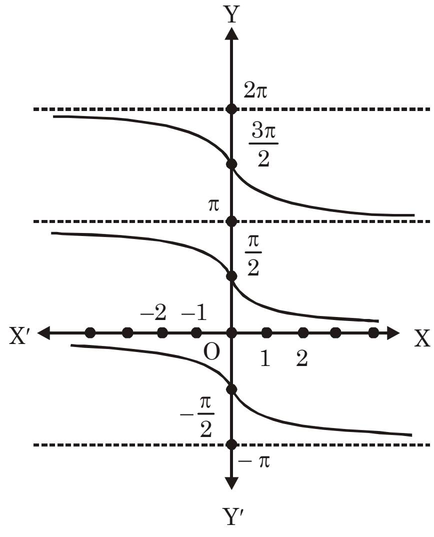
$y = \sec^{-1} x$

- **Function:** $y = \tan^{-1}x$
  Domain: R
  Range: $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$



$y = \tan^{-1} x$

- **Function:** $y = \cot^{-1}x$
  Domain: R
  Range: (0, π)



$y = \cot^{-1} x$

- **Properties:**
  - $\sin^{-1}(\sin x) = x$
  - $\cos^{-1}(\cos x) = x$
  - $\tan^{-1}(\tan x) = x$
  - $\text{cosec}^{-1}(\text{cosec } x) = x$
  - $\sec^{-1}(\sec x) = x$
  - $\cot^{-1}(\cot x) = x$
  - $\sin^{-1}(x) = \text{cosec}^{-1}\left(\frac{1}{x}\right),\ x \in [-1, 1]$
  - $\text{cosec}^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right),\ x \in (-\infty, -1] \cup [1, \infty)$
  - $\cos^{-1}(x) = \sec^{-1}\left(\frac{1}{x}\right),\ x \in [-1, 1]$
  - $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right),\ x \in (-\infty, -1] \cup [1, \infty)$
  - $\tan^{-1}(x) = \begin{cases} \cot^{-1}\left(\frac{1}{x}\right), & x > 0 \\ -\pi + \cot^{-1}\left(\frac{1}{x}\right), & x < 0 \end{cases}$
  - $\cot^{-1}(x) = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right), & x > 0 \\ \pi + \tan^{-1}\left(\frac{1}{x}\right), & x < 0 \end{cases}$

- $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2},\ x \in [-1, 1]$
- $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2},\ x \in R$
- $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2},\ |x| \geq 1$
- $\sin^{-1}(-x) = -\sin^{-1} x$
- $\cos^{-1}(-x) = \pi - \cos^{-1} x$
- $\tan^{-1}(-x) = -\tan^{-1} x$
- $\cot^{-1}(-x) = \pi - \cot^{-1} x$
- $\sec^{-1}(-x) = \pi - \sec^{-1} x$
- $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$
- $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$
- $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}$
- $2\tan^{-1} x = \begin{cases} \sin^{-1}\left(\frac{2x}{1+x^2}\right), & |x| \leq 1 \\ \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), & |x| \geq 0 \\ \tan^{-1}\left(\frac{2x}{1-x^2}\right), & -1 < x < 1 \end{cases}$

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

1 Mark Questions

1. Write the principal value of

$\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right)$.

[ALL INDIA 2013]

2. Write the value of $\tan\left(2\tan^{-1}\frac{1}{5}\right)$.

[DELHI 2013]

3. Write the principal value of

$\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(\frac{1}{2}\right)$.

[DELHI 2012]

4. Write the value of $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right]$

[DELHI 2011]

5. Write the principal value of $\tan^{-1}(-1)$

[ALL INDIA 2011]

6. Write the principal value of

$\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$.

[ALL INDIA 2013, 2018]

7. Write the value of $\tan^{-1}\left[2\sin\left[2\cos^{-1}\frac{\sqrt{3}}{2}\right]\right]$.

[All India 2013]

8. If $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}, xy < 1,$ then write the value of $x + y + xy$.

[ALL INDIA 2014]

4 Marks Questions

9. Find the value of the following:

$$\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right], |x| < 1, y > 0$$

and $xy < 1$.

[DELHI 2013]

10. If $\sin\left[\cot^{-1}(x+1)\right] = \cos\left(\tan^{-1} x\right)$, then find x.

[DELHI 2015]

11. Prove that

$$\tan\left\{\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} + \tan\left\{\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} = \frac{2b}{a}.$$

[DELHI 2017]

[TOPIC 1] Matrix and Operations on Matrices

Summary

- A **matrix** is an ordered rectangular array of numbers or functions. The numbers are called the **elements** of the matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & \dots & a_{3n} \\ . & . & . & \dots & \dots & . \\ . & . & . & \dots & \dots & . \\ a_{m1} & a_{m2} & a_{m3} & \dots & \dots & a_{mn} \end{bmatrix}_{m\times n}$$

Example: $A = \begin{bmatrix} 4 & \frac{1}{2} & -2 \\ 0 & 3 & 5 \\ 1 & \sqrt{6} & -7 \end{bmatrix}$

- The order of the matrix is determined by $m \times n$ where m is the number of rows and n is the number of columns.
- The matrix with $m \times n$ order can be represented as $A = \left[a_{ij}\right]_{m\times n}$; $i, j \notin \mathbb{N}$ also $1 \le i \le m, 1 \le j \le n,$.

Types of Matrices

- **Column matrix** is a matrix which has only 1 column. It is defined as $A = [a_{ij}]_{m\times 1}$.

Example: $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}_{3\times 1}$

- **Row matrix** is a matrix which has only row. It is defined as $B = [b_{ij}]_{1\times n}$.

Example: $\begin{bmatrix} \sqrt{2} & -1 & 3 \end{bmatrix}_{1\times 3}$

- **The square matrix** is the matrix which has equal number of rows and columns i.e. the matrix in which $m = n$. It is defined as $A = [a_{ij}]_{m\times m}$.

Example: $\begin{bmatrix} 3 & 9 & 1 \\ 7 & 6 & 3 \\ 9 & 0 & 2 \end{bmatrix}_{3\times 3}$

The square matrix $A = [a_{ij}]_{m\times m}$ is said to be a diagonal matrix if all its non-diagonal elements are zero. It is defined as $A = [a_{ij}]_{m\times m}$ if $a_{ij} = 0$, when $i \ne j$.

- A **scalar matrix** is the one in which the diagonal elements of a diagonal matrix are equal. It is defined as $A = [a_{ij}]_{m\times m}$ if $a_{ij} = 0$, when $i \ne j$ and $a_{ij} = k$, when $i = j$, where k is some constant.

Example: $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

- **Identity matrix** is the square matrix where the diagonal elements are all 1 and rest are all zero. It is defined as $A = [a_{ij}]_{m\times m}$ where $a_{ij} = 1$ if $i = j$ and $a_{ij} = 0$ if $i \ne j$.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

- **A zero matrix** is the one in which all the elements are zero.

Example: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$

- Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if they are of the same order and also each element of matrix A is equal to the corresponding element of Matrix B.
- The **sum of the two matrices** $A = [a_{ij}]$ and $B = [b_{ij}]$ of same order $m \times n$ is defined as $C = [c_{ij}]_{m\times n}$where $c_{ij} = a_{ij} + b_{ij}$.
- If x is a scalar and $A = [a_{ij}]_{m\times n}$ is a matrix, then xA is the matrix obtained by multiplying each element of the matrix by the scalar x. It can be defined as $xA = x[a_{ij}]_{m\times m} = [x(a_{ij})]_{m\times n}$.
- $-A$ denotes the negative of a matrix. $-A = (-1)A$.
- The **difference of the two matrices** $A = [a_{ij}]$ and $B = [b_{ij}]$of same order $m \times n$ is defined as $C = [c_{ij}]_{m\times n}$ where $c_{ij} = a_{ij} - b_{ij}$.
- If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same order, say $m \times n$ then $A + B = B + A$. It is called the **commutative law**.

- If $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ are the three matrices of same order, say $m \times n$, then $(A + B) + C = A + (B + C)$. It is called the **associative law**.
- If $B = [b_{ij}]$ is a matrix of order $m \times n$ and O is a zero matrix of the order $m \times n$, then $B + O = O + B = B$. O is the additive identity for matrix addition.
- If $B = [b_{ij}]$ is a matrix of order $m \times n$ then we have another matrix as $-B = [-b_{ij}]$ of the order $m \times n$ such that $B + (-B) = (-B) + B = O$. So, $-B$ is the additive inverse of B.
- If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same order, say $m \times n$, and x and y are the scalars, then
  - ➢ $x(A + B) = xA + xB$
  - ➢ $(x + y)A = xA + yA$
- If $A = [a_{ij}]$ is a matrix of order $m \times n$ and $B = [b_{jk}]$ is a matrix of order $n \times p$ then the product of the matrices A and B is a matrix C of order $m \times p$. It can be denoted as $AB = C = [C_{ik}]_{m\times p}$, where

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

- **Properties of multiplication of matrices** are as follows:
  - ➢ The associative law: If there are 3 matrices X, Y and Z we have $(XY)Z = X(YZ)$
  - ➢ Distributive law: If there are 3 matrices X, Y and Z then:

    $X(Y + Z) = XY + XZ$

    $(X + Y)Z = XZ + YZ$
  - ➢ The existence of multiplicative identity: For every square matrix X, there exists an identity matrix of the same order such that $IX = XI = X$.
  - ➢ Multiplication is not commutative: $AB \neq BA$

PREVIOUS YEARS'
EXAMINATION QUESTIONS
TOPIC 1
1 Mark Questions

1. Find the value of a if

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}.$$

[DELHI 2013]

2. If $\begin{bmatrix} x+1 & x-1 \\ x-3 & x+2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}$, then write the value of x.

[DELHI 2013]

3. If $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$, then find the matrix A.

[DELHI 2013]

4. Write the element a_{23} of a 3×3 matrix $A = (a_{ij})$ whose elements a_{ij} are given by $a_{ij} = \frac{|i-j|}{2}$.

[DELHI 2015]

5. If A is a 3×3 invertible matrix, then what will be the value of k if

$$\det\left(A^{-1}\right) = (\det A)^k .$$

[DELHI 2017]

6. If A is a square matrix such that $A^2 = I$, then find the simplified value of

$$(A-I)^3 + (A+I)^3 - 7A$$

[DELHI 2016]

7. If $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$, write the value of x.

[ALL INDIA 2012]

8. For a 2 × 2 matrix, A = $\left[a_{ij}\right]$, whose elements are given by $a_{ij} = \frac{i}{j}$, write the value of a_{12}

[DELHI 2011]

9. For what value of x the matrix $\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$ is singular?

[DELHI 2011]

[TOPIC 2] Transpose of a Matrix and Symmetric and Skew Symmetric Matrices

Summary

- The **transpose of the matrix** $A = [a_{ij}]_{m\times n}$ is denoted by $A^T = [a_{ji}]_{n\times m}$ and is obtained by interchanging the rows with columns of matrix A.

  Example: If $A=\begin{bmatrix}-4 & 1\\ 2 & 0\end{bmatrix}$, then $A^T=\begin{bmatrix}-4 & 2\\ 1 & 0\end{bmatrix}$.
- Some properties of transpose of the matrices are as follows:

  If A and B are matrices of suitable orders then
  - $(A^T)^T = A$
  - $(kA)^T = kA^T$ (Where k is any constant)
  - $(A+B)^T = A^T + B^T$
  - $(AB)^T = B^T A^T$
- If $A^T = A$ then the square matrix $A = [a_{ij}]$ is said to be **symmetric matrix** for all possible values of i and j.

  Example: $\begin{bmatrix}1 & 2 & 4\\ 2 & 3 & 7\\ 4 & 7 & 0\end{bmatrix}$
- If $A^T = -A$ then the square matrix $A = [a_{ij}]$ is said to be **skew symmetric matrix** for all the possible values of i and j. All the diagonal elements of a skew symmetric matrix are zero.

  Example: $A=\begin{bmatrix}1 & 2 & -4\\ -2 & 3 & 7\\ 4 & -7 & 0\end{bmatrix}$ then

  $$A^T=\begin{bmatrix}1 & -2 & 4\\ 2 & 3 & -7\\ -4 & 7 & 0\end{bmatrix}=-\begin{bmatrix}1 & 2 & -4\\ -2 & 3 & 7\\ 4 & -7 & 0\end{bmatrix}=-A$$
- For any square matrix A with real number entries, $A + A^T$ is a symmetric matrix and $A - A^T$ is a skew symmetric matrix.
- Any square matrix can be expressed as the sum of a symmetric matrix and a skew symmetric matrix

  i.e $A=\frac{1}{2}\left(A+A^T\right)+\frac{1}{2}\left(A-A^T\right)$

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 2

1 Mark Questions

1. Matrix $A=\begin{vmatrix}0 & 2b & -2\\ 3 & 1 & 3\\ 3a & 3 & -1\end{vmatrix}$ is given to be symmetric, find values of a and b.

[DELHI 2016]

2. If the matrix $A=\begin{bmatrix}0 & a & -3\\ 2 & 0 & -1\\ b & 1 & 0\end{bmatrix}$ a skew symmetric matrix, find the value of 'a' and 'b'?

[DELHI 2018]

3. For what value of x, is the matrix $A=\begin{bmatrix}0 & 1 & -2\\ -1 & 0 & 3\\ x & -3 & 0\end{bmatrix}$ a symmetric matrix?

[ALL INDIA 2013]

4. If A is a skew-symmetric matrix of order 3, then prove that det A = 0.

[ALL INDIA 2017]

5. If A = $\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}$, find $0<\alpha<\frac{\pi}{2}$ satisfying $A+A^T=\sqrt{2}I_2$ when A^T is transpose of A.

[ALL INDIA 2016]

2 Marks Question

6. Show that all the diagonal elements of a skew symmetric matrix are zero.

[DELHI 2017]

[TOPIC 3] Inverse of matrices by Elementary row transformation

- Elementary operation of a matrix are as follows:
  - **Interchanging of two rows or columns:** $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$ represents that the ith row or column is interchanged with jth row or column.
  - **Multiplying the row or column of matrix by non- zero scalar:** $R_i \to lR_j$ or $C_i \to lC_j$ where I is any non- zero number, represents the ith row or column is multiplied by I.
  - **Adding the elements of any row or column to another row or column:** $R_i \to R_i + lR_j$ or $C_i \to C_i + lC_j$, where I is any non-zero number, represents that the jth row or column is multiplied by I and added to respective element of ith row or column.
- If X is a square matrix of order n and if there exists another square matrix Y of the same order n, such that $XY = YX = I$, then Y is called the inverse matrix of X and it is denoted by X^{-1}.
- The inverse of a matrix can be found using row or column operations.
- If Y is the inverse of X, then X is also the inverse of Y. Also, $(XY)^{-1} = Y^{-1}X^{-1}$
- Inverse of a square matrix, if it exists, is unique.

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 3

1 Mark Question

1. If $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$, write the minor of the element a_{23}.

[DELHI 2012]

6 Marks Questions

2. Use elementary transformations, find the inverse of the matrix $A = \begin{pmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

And use it to solve the following system of linear equations:

$8x + 4y + 3z = 19 \quad 2x + y + z = 5 \quad x + 2y + 2z = 7$

[DELHI 2016]

3. Using elementary transformations, find the inverse of the matrix $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

[DELHI 2011]

4. Using elementary operations, find the inverse of the following matrix:

$$\begin{pmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$$

[DELHI 2012]

5. Using elementary row transformations, find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$.

[DELHI 2018]

6. Find A-1 using row elementary operations, given that $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$.

[DELHI 2011]

[TOPIC 1] Expansion of Determinants

Summary

- **Definition**: A determinant is a number (real or complex) that can be related to any square matrix A = $[a_{ij}]$ of order n. It is denoted as det(A).

  Determinant of a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ can be given as:

  $$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

  Determinant of a matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ by expanding along R_1 can be given as:

  $$\det(A) = |A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- **Minors:** The minor M_{ij} of a_{ij} in A is the determinant of the square sub matrix of order $(n-1)$ obtained by deleting its i^{th} row and j^{th} column in which a_{ij} lies. It is denoted by M_{ij}.

  Minor of an element of a determinant of order n(for all $n \geq 2$) is a determinant of order $n - 1$.

- **Co-factors:** Co-factor of an element a_{ij} is defined by $A_{ij} = (-1)^{(i+j)} M_{ij}$, where M_{ij} is a minor of a_{ij}. Co-factor is denoted by A_{ij}.

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

1 Mark Questions

1. If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, Find the value of x.

[ALL INDIA 2014]

2. If A_{ij} is the cofactor of element a_{ij} of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$, then write the value of $a_{32} . A_{32}$.

[ALL INDIA 2013]

3. $\begin{vmatrix} x & x \\ 1 & x \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$, write the positive value of x

[ALL INDIA 2011]

4. If $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$, write the minor of the elements a_{23}.

[ALL INDIA 2012]

5. Simplify:

$$\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

[DELHI 2012]

6. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, write the value of x.

[DELHI 2014]

7. Find the maximum value of

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$$

[DELHI 2016]

[TOPIC 2] Properties of Determinants

Summary

- **Properties of determinants**:
  - ➢ The value of the determinant remains unchanged if its rows and columns are interchanged.
  - ➢ If A is a square matrix, then $det(A) = det(A')$, where A'= transpose of A.
  - ➢ If any two columns (or rows) of a determinant are interchanged, then the sign of the determinant changes.
  - ➢ The determinant of the product of two square matrices of same order is equal to the product of their respective determinants, that is $|AB| = |A||B|$.
  - ➢ If any two rows or columns of a determinant are identical (i.e. all corresponding elements are same), then value of determinant is zero.
  - ➢ If each element of a row or a column of a determinant is multiplied by a constant k, then its value gets multiplied by k.
  - ➢ If some or all elements of a row or column of a determinant are expressed as sum of two or more terms, then the determinant can be expressed as sum of two or more determinants.

    For example:

$$\begin{vmatrix} a_1+x & a_2+y & a_3+z \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

  - ➢ If we multiply each element of a row or a column of a determinant, by a constant k, then the value of the determinant is also multiplied by k.
  - ➢ Multiplying a determinant by k means multiply elements of any one row or any one column by k.
  - ➢ Adding or subtracting each element of anycolumn or any row of a determinant withthe equimultiples of corresponding elements of any other row or column, does not changes the value of the determinant, i.e., the value of determinant remain same if we apply the operation $R_i \to R_i \to kR_j$.
- **Area of triangle:** If a triangle is given with its vertices at points (x_1, y_1), (x_2, y_2) and (x_3, y_3), then its area can be calculated as:

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

  - ➢ Area is a positive quantity, so we always take the absolute value of the determinant.
  - ➢ If the area of the triangle is already given, use both positive and negative values of the determinant for calculation.
  - ➢ Area of triangle formed by three collinear points is always zero.

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 2

1 Mark Question

1. If A is a 3 × 3 matrix and |3A| = k|A|, then write the value of k.

[ALL INDIA 2016]

4 Marks Questions

2. Using the properties of determinants, prove the following :

$$\begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(1-x) & x(x-1)(x-2) & x(x+1)(x-1) \end{vmatrix}$$

$$= 6x^2(1-x^2)$$

[ALL INDIA 2015]

[TOPIC 3] Adjoint and Inverse of a Matrix

Summary

- **Adjoint of a square matrix:**

  The adjoint of a square matrix $A = [a_{ij}]_{n\times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n\times n}$, where A_{ij} is the cofactor of the element a_{ij}. Adjoint of the matrix A is denoted by "$adj(A)$".

  Example: $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ then, $adj\,A = \begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix}$

  If A be any given square matrix of order n, then $A(adj\,A) = (adj\,A)A = AI$, where I is the identity matrix of order n.

- **Inverse of a Matrix**
  - **Singular matrix:** It is a matrix with zero determinant value. i.e. $|A| = 0$.
  - **Non-singular matrix:** It is a matrix with a non-zero determinant value. i.e. $|A| \neq 0$.

    If A and B are nonsingular matrices of the same order, then AB and BA are also nonsingular matrices of the same order.
  - A square matrix A is invertible if and only if A is nonsingular matrix.
  - If A is a nonsingular matrix, then its **inverse** exists which is given by $A^{-1} = \dfrac{1}{|A|} adj(A)$
- **Consistent system:** A system of equations is said to be consistent if there exist one or more solution to the system of equation.
- **Inconsistent system:** If the solution to the system of equation does not exist, then it is termed as inconsistent system.
- For the square matrix A in the matrix equation $AX = B$
  - $|A| \neq 0$, there exists unique solution. The system of equation is consistent.
  - $|A| = 0$ and $(adj\,A)B \neq 0$ then there exists no solution. The system is inconsistent.
  - $|A| = 0$ and $(adj\,A)B = 0$, then system may or may not be consistent.

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 3

1 Mark Questions

1. If for any 2×2 square matrix A, A(adj A) $= \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$, then write the value of $|A|$.

   [ALL INDIA 2017]

2. Write A^{-1} for $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$.

   [DELHI 2011]

3. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, then write A^{-1}.

   [ALL INDIA 2015]

4. For what values of k, the system of linear equations

   $x + y + z = 2$

   $2x + y - z = 3$

   $3x + 2y + kz = 4$ has a unique solution ?

   [ALL INDIA 2016]

2 Marks Question

5. Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1} and show that $2A^{-1} = 9I - A$.

   [DELHI 2018]

4 Marks Questions

6. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$, find $(A')^{-1}$.

   [DELHI 2015]

7. If $A = \begin{vmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{vmatrix}$, find $A^2 - 5A + 16I$.

   [ALL INDIA 2015]

8. If $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$ and $A^3 - 6A^2 + 7A + kI_3 = 0$.

   Find k.

   [ALL INDIA 2016]

[TOPIC 1] Continuity

Summary

- **Definition of Continuity:**

  Let f is a real valued function and is a subset of real numbers and a point c lies in the domain of f, then f is continuous at c if

  $$\lim_{x \to c} f(x) = f(c)$$

  When the function f is discontinuous at c, it is called the point of discontinuity of f.

  Also, if f is defined on $[a, b]$ then continuity of a function f at a means

  $$\lim_{x \to a^+} f(x) = f(a)$$

  And continuity of the function f at b means

  $$\lim_{x \to b^-} f(x) = f(b)$$

- Every polynomial function is continuous.
- Consider two real functions f and g which are continuous at c, then sum, difference, product and quotient of the two functions will also be continuous at $x = c$.

  i.e. $(f + g)(x) = f(x) + g(x)$ is continuous at $x = c$

  $(f - g)(x) = f(x) - g(x)$ is continuous at $x = c$

  $(f \,.\, g)(x) = f(x) \,.\, g(x)$ is continuous at $x = c$

  Here if f is a constant function say $f(x) = \alpha$ for some real number α, then the function $(\alpha \,.\, g)$ defined by $(\alpha \,.\, g)(x) = \alpha \,.\, g(x)$ is also continuous. If $\alpha = -1$ then continuity of f implies continuity of $-f$.

  $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ is continuous at $x = c$ when $g(x) \neq 0$

  Here, if f is a constant function say $f(x) = \alpha$ for some real number α, then the function $\dfrac{\alpha}{g}$ defined by $\dfrac{\alpha}{g}(x) = \dfrac{\alpha}{g(x)}$ is also continuous wherever $g(x) \neq 0$.

- **Here are some formulae for limits:**

  - $\lim\limits_{x \to 0} \cos x = 1$
  - $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$
  - $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1$
  - $\lim\limits_{x \to 0} \dfrac{\sin^{-1} x}{x} = 1$
  - $\lim\limits_{x \to 0} \dfrac{\tan^{-1} x}{x} = 1$
  - $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \log_e a,\ a > 0$
  - $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$
  - $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$
  - $\lim\limits_{x \to 0} \dfrac{\log_e(1 + x)}{x} = 1$
  - $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$

[TOPIC 2] Differentiability

Summary

- **Differentiability:**

  Consider a real function f and a point c lies in its domain then the derivative of that function at c is defined by

  $$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

  Provided the limit exists. It is denoted by $f'(c)$ or $\frac{d}{dx}(f(x))$.

- **Some rules for algebra of derivatives:**
  - ➢ $(u + v)' = u' + v'$
  - ➢ $(u - v)' = u' - v$
  - ➢ Product rule: $(uv)'$ $u'v + uv'$
  - ➢ Quotient rule: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, $v \neq 0$

- A function which is differentiable at a point c is also continuous at that point but the converse is not true.

- **Chain Rule:**

  Consider a real value function f which is a composite of u and v.

  Let $t = u(x)$ and $\frac{dt}{dx}, \frac{dv}{dt}$ exists, then $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$

- **Some important features of exponential function and logarithm function are given below**
  - ➢ Domain of both the functions is set of all real numbers.
  - ➢ Range of exponential function is set of all positive real numbers and the range of log function is set of all real numbers.
  - ➢ The point (0, 1) is always on the graph of exponential function and the point (1, 0) is always on the graph of log function.
  - ➢ Both the functions are ever increasing.

- A relation expressed between two variable x and y in the form $x = f(t)$, $y = g(t)$ is said to be **parametric form** with t as a parameter.

  By using Chain Rule we find the derivative of function in such form.

  $$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

  or $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \left(\text{whenever } \frac{dx}{dt} \neq 0\right)$$

  Thus $$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$$

- **Second Order Derivative:**

  If $y = f(x)$

  $$\frac{dy}{dx} = f'(x)$$

  If $f'(x)$ is differentiable then $\frac{dy}{dx} = f'(x)$ will be differentiated again. The left side will become $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ and is called second order derivative of y w.r.t. x.

- **Rolle's Theorem:**

  Consider a real valued function f defined on the interval $[a, b]$ such that the function is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$, then there exist a point c in (a, b) such that $f'(c) = 0$.

- **Lagrange's Mean Value Theorem:**

  Consider a real valued function f defined on the interval $[a, b]$ such that the function is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

- **Derivatives of some standard functions are listed below:**
  - ➢ $\frac{d}{dx}(x^n) = nx^{n-1}$
  - ➢ $\frac{d}{dx}(k) = 0$, k is any constant
  - ➢ $\frac{d}{dx}(a^x) = a^x \log_e a$, $a > 0$
  - ➢ $\frac{d}{dx}(e^x) = e^x$
  - ➢ $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a} = \frac{1}{x} \log_a e$

- $\frac{d}{dx}(\log_e x) = \frac{1}{x}$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\cot x) = -\cos ec^2 x$
- $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$
- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \ x \in (-1, 1)$
- $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \ x \in (-1, 1)$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \ x \in \mathbb{R}$
- $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, \ x \in \mathbb{R}$
- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$, where $x \in (-\infty, -1) \cup (1, \infty)$
- $\frac{d}{dx}(\text{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$, where $x \in (-\infty, -1) \cup (1, \infty)$

PREVIOUS YEARS' EXAMINATION QUESTIONS

TOPIC 2

2 Marks Question

1. Find $\frac{dy}{dx}$ at $x = 1, \ y = \frac{\pi}{4}$ if $\sin^2 y + \cos xy = K$

[DELHI 2017]

4 Marks Questions

2. If $x^y + y^x = a^b$, then find $\frac{dy}{dx}$

[ALL INDIA 2017]

3. If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), x^2 \le 1,$ then find $\frac{dy}{dx}$

[DELHI 2015]

4. $e^y(x+1) = 1$ then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

[ALL INDIA 2017]

5. If x cos(a + y) = cos y then prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

Hence show that $\sin a \frac{d^2y}{dx^2} + \sin 2(a+y)\frac{dy}{dx} = 0$

[ALL INDIA 2016]

6. **Find** $\frac{dy}{dx}$ **if** $y = \sin^{-1}\left[\frac{6x - 4\sqrt{1-4x^2}}{5}\right]$

[ALL INDIA 2016]

7. Differentiate the following function with respect to x: $(\log x)^x + x^{\log x}$

[DELHI 2013]

8. If $y = \log\left[x + \sqrt{x^2+a^2}\right]$, show that $(x^2+a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

[DELHI 2013]

[TOPIC 1] Rate of Change, Increasing and Decreasing Functions and Approximations

Summary

- **Rate of Change of Bodies** representing $\frac{dy}{dx}$ as a rate measure:

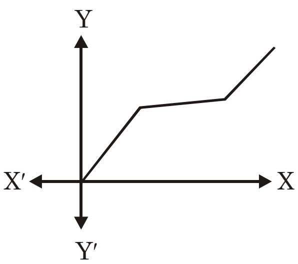
  Let us take two variables x and y that vary with respect to another variable says, i.e. if $x = f(s)$ and $y = g(s)$, then by applying the chain rule, we have

  $$\frac{dy}{dx} = \frac{\frac{dy}{ds}}{\frac{dx}{ds}}, \text{ if } \frac{dx}{ds} \neq 0.$$

  Thus, the rate of change of y with respect to x can be calculated using the rate of change of y and that of x both with respect to s.

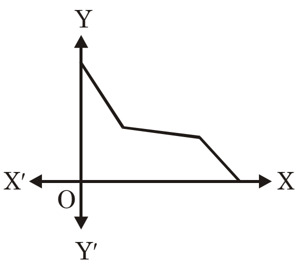
- **Increasing and decreasing functions**
  - ➢ A function is said to be increasing when the y value increases as the x value increases.

    Example:



  - ➢ A function is said to be increasing when the y value decreases as the x value increases.

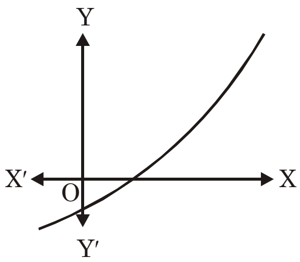
    Example:



  - ➢ f is strictly increasing if

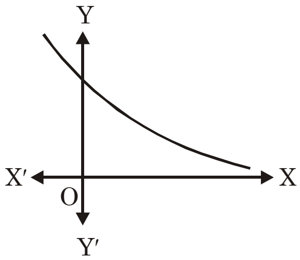
    $$x_1 < x_2,$$
    $$\Rightarrow \quad f(x_1) < f(x_2).$$

    Example:



  - ➢ f is strictly decreasing if $x_1 < x_2, \Rightarrow f(x_1) < f(x_2)$.

    Example:



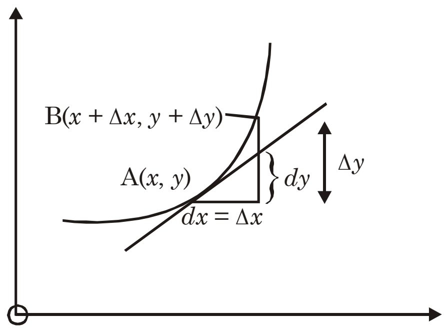
  - ➢ Let f be continuous on [a, b] and differentiable on the open interval (a, b). Then:

    f is increasing in [a, b] if $f'(x) > 0$ for each $x \in (a, b)$

    f is decreasing in [a, b] if $f'(x) < 0$ for each $x \in (a, b)$

    f is a constant function in [a, b] if $f'(x) = 0$ for each $x \in (a, b)$

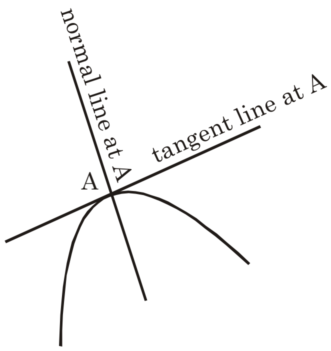
- **Approximations**
  - ➢ Let the given function be $y = f(x)$. Δx denotes a small increment in x.
  - ➢ The corresponding increment in y is given by $\Delta y = f(x + \Delta x) - f(x)$
  - ➢ Differential of y, denoted by dy is $dy = \left(\frac{dy}{dx}\right)\Delta x$



[TOPIC 2] Tangents and Normals

Summary

- A tangent line is defined as a straight line that touches the given function at only one point and it represents the instantaneous rate of change of function at the point.
- A normal line to a point (x, y) on a curve is the line that goes through the point (x, y) and is perpendicular to the tangent line.



- Slope or gradient of a line: If a line makes an angle θ with the positive direction of X axis in anti-clockwise direction, then $\tan\theta$ is called the slope or gradient of the line.
- If a tangent line to the curve y = f(x) makes an angle θ with x-axis in the positive direction, then

  slope of the tangent = $\tan\theta = \dfrac{dy}{dx}$
- If slope of the tangent line is zero, then $\tan\theta = 0$ and so $\theta = 0$ which means the tangent line is parallel to the x-axis. In this case, the equation of the tangent at the point is given by $(y = y_0)$
- If $\theta \to \dfrac{\pi}{2}$ then $\tan\theta \to \infty$, which means the tangent line is perpendicular to the x-axis, i.e., parallel to the y-axis. In this case, the equation of the tangent at (x_0, y_0) is given by $(x = x_0)$
- Equation of tangent at (x_1, y_1) is given by $(y - y_1) = m_T(x - x_1)$, where m_T is the slope of the tangent such that $m_N = \left[\dfrac{dy}{dx}\right]_{(x_1, y_1)}$
- Equation of normal at (x_1, y_1) is given by $(y - y_1) = m_N(x - x_1)$, where m_N is the slope of the normal such that $m_N = \dfrac{-1}{\left[\dfrac{dy}{dx}\right]_{(x_1, y_1)}}$
- Tangent and normal are perpendicular to each other, which gives us $m_T \times m_N = -1$
- If the slope of two different curves are m_1 and m_2, then the acute angle between them is given by $\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 . m_2}\right|$
- The slope intercept form of the line is $y = mx + c$, where m is the slope of the given line.

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 2

4 Marks Questions

1. Show that the equation of normal at any point on the curve
   $x = 3\cos t - \cos^3 t$ and $y = 3\sin t - \sin^3 t$ is $4(y\cos^3 t - x\sin^3 t) = 3\sin 4t$

   [DELHI 2016]

2. Find the equations of the tangent and normal to the curve $x = a\sin^3\theta$ and $y = a\cos^3\theta$ at $\theta = \dfrac{\pi}{4}$.

   [DELHI 2014]

3. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to x-axis.

   [DELHI 2011]

4. Find the equations of the tangent and the normal, to the curve $16x^2 + 9y^2 = 145$ at the point (x_1, y_1), where $x_1 = 2$ and $y_1 > 0$.

   [DELHI 2018]

[TOPIC 3] Maxima and Minima

Summary

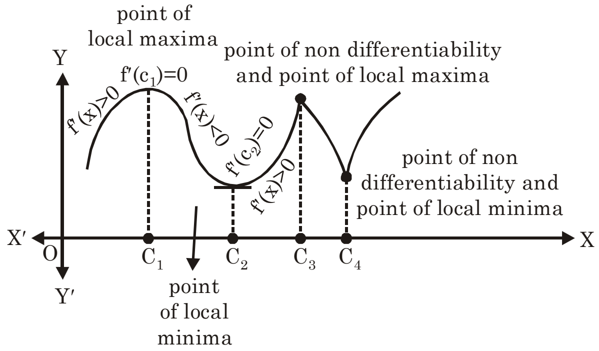
- The maximum value attained by a function is called maxima and the minimum value attained by the function is known as minima.
- Consider $y = f(x)$ be a well-defined function on an interval I, then
  - ➢ f is said to have a maximum value in I, if there exist a point c in I such that $f(c) > f(x)$, $\forall x \in$ I. The value corresponding to f(c) is called as maximum value of x in I and the point c is the maximum value.
  - ➢ f is said to have a minimum value in I, if there exist a point c in I such that $f(c) < f(x)$, $\forall x \in I$. The value corresponding to f(c) is called as minimum value of x in I and the point c is the minimum value.
  - ➢ f is said to have an extreme value in I, if there exist a point c in I such that f(c) is either a maximum value or a minimum value. The value corresponding to f(c) is called as extreme value of x in I and the point c is the extreme point.
- Let f be a function defined on an open interval I. Suppose c ? I be any point. If f has a local maxima or a local minima at x = c, then either $f'(c) = 0$ or f is not differentiable at c.

First Derivative Test

Let f be a function defined on an open interval I.

Let f be continuous at a critical point c in I. Then

- If $f'(x)$ changes sign from positive to negative as x increases through c, i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of c, and $f'(x) < 0$at every point sufficiently close to and to the right of c, then c is a point of local maxima.
- If $f'(x)$ changes sign from negative to positive as x increases through c, i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of c, and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of local minima.
- If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. In fact, such a point is called point of inflection.



Second Derivative Test

Let f be a function defined on an interval I and c $\in$ I. Let f be twice differentiable at c. Then

- x = c is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$. The value f (c) is local maximum value of f.
- x = c is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$ In this case, f (c) is local minimum value of f.
- The test fails if $f'(c) = 0$ and $f''(c) = 0$. In this case, we go back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflexion.
- **Maximum and Minimum values of a function in a closed interval**

  Let f be a continuous function on an interval I = [a, b]. Then f has the absolute maximum value and f attains it at least once in I. Also, f has the absolute minimum value and attains it at least once in I.

  Let f be a differentiable function on a closed interval I and let c be any interior point of I. Then

  - ➢ $f'(c) = 0$ if f attains its absolute maximum value at c.
  - ➢ $f'(c) = 0$ if f attains its absolute minimum value at c.

  In view of the above results, we have the following working rule for finding absolute maximum and/ or absolute minimum values of a function in a given closed interval [a, b].

- **Working Rule**
  - ➢ Find all critical points of f in the interval, i.e., find points x where either f ′(x) = 0 or f is not differentiable.
  - ➢ Take the end points of the interval.
  - ➢ At all these points (listed in Step 1 and 2), calculate the values of f
  - ➢ Identify the maximum and minimum values of f out of the values calculated in
  - ➢ This maximum value will be the absolute maximum (greatest) value off and the minimum value will be the absolute minimum (least) value of f.

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 3

4 Marks Questions

1. If the function $f(x)=2x^3-9mx^2+12m^2x+1$, where $m>0$ attains its maximum and minimum at p and q respectively such that $p^2=q$, then find the value of m.

[ALL INDIA 2015]

2. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

[ALL INDIA 2011]

6 Marks Questions

3. Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube.

[ALL INDIA 2017]

4. The volume of a cube is increasing at the rate of 9 cm^3/s. How fast is its surface area increasing when the length of an edge is 10 cm ?

[ALL INDIA 2017]

5. If the sum of lengths of the hypotenuse and a side of a right angles triangle is given, show that the area of the triangle is maximum, when the angle between them is $\frac{\pi}{3}$.

[DELHI 2017]

6. Show that the semivertical angle of the cone of the maximum volume and of given slant height is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

[DELHI 2014]

7. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also find the maximum volume in terms of volume of the sphere.

[ALL INDIA 2014]

8. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area,

[DELHI 2011]

9. Show that the height of a closed right circular cylinder of given surface and maximum volume, is equal to the diameter of its base.

[DELHI]

10. Show that semi-vertical angle of a cone of maximum volume and given slant height is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

[ALL INDIA 2016]

11. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is 4r/3, Also show that the maximum volume of the cone is 8/27 of the volume of the sphere.

[DELHI 2014]

[TOPIC 1] Indefinite Integrals

Summary

- Integration is the inverse of differentiation. Instead of differentiating a function, we will be given the derivative of a function and we would be asked to find its primitive function. Such a process is called integration or anti differentiation.

  If $\frac{d}{dx}F(x) = f(x)$. Then we write $\int f(x)dx = F(x) + C$.

  These integrals are called indefinite integrals or general integrals and C is called the constant of integration.
- The integral of a function are unique upto an additive constant, i.e. any two integrals of a function differ by a constant.
- When a polynomial function P is integrated, the result is a polynomial whose degree is one more than that of P.
- Geometrically, the indefinite integral of a function represents a family of curves placed parallel to each other having parallel tangents at the points of intersection of the curves of the family with the lines perpendicular to the axisrepresenting the variable of integration.
- Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent.
- Some properties of indefinite integral are as follows:
  - ➢ $\int [f(x) + g(x)]\,dx = \int f(x)dx + \int g(x)dx$
  - ➢ For any real number a, $\int af(x)dx = a\int f(x)dx$
  - ➢ Properties (i) and (ii) can be generalised to a finite number of functions $f_1, f_2, f_3, \ldots f_n$ and the real numbers, $a_1, a_2, a_3, \ldots a_n$ giving

$$\int a_1 f_1(x)\mathrm{dx} + a_2 f_2(x)\mathrm{dx} + \ldots + \mathrm{a}_n f_n(x)dx$$

$$= a_1\int f_1(\mathrm{x})\mathrm{dx} + \mathrm{a}_2\int f_2(x)\mathrm{dx} + \ldots \mathrm{a}_n\int f_n(x)dx.$$

- Some basic integrals are as follows:
  - ➢ $\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad , n \neq -1$
  - ➢ $\int dx = x + C$
  - ➢ $\int \cos x dx = \sin x + C$
  - ➢ $\int \sin x dx = -\cos x + C$
  - ➢ $\int \sec^2 x dx = \tan x + C$
  - ➢ $\int \operatorname{cosec}^2 x dx = -\cot x + C$
  - ➢ $\int \sec x \tan x dx = \sec x + C$
  - ➢ $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$
  - ➢ $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
  - ➢ $\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$
  - ➢ $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
  - ➢ $\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$
  - ➢ $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$
  - ➢ $\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + C$
  - ➢ $\int e^x dx = e^x + C$
  - ➢ $\int \frac{1}{x} dx = \log|x| + C$
  - ➢ $\int a^x dx = \frac{a^x}{\log a} + C$
- **Integration by substitution:**

  The method in which we change the variable to some other variable is called the method of substitution. It often reduces an integral to one of the fundamental integrals.

  The integral $\int f(x)dx$ can be substituted into another form by changing the independent variable x to t by substituting $x = g(t)$.

Consider, $A = \int f(x)dx$

Put $x = g(t)$, therefore $\frac{dx}{dt} = g'(t)$.

$\Rightarrow dx = g'(t)dt$

Thus, $A = \int f(x)\,dx = \int f(g(t))\,g'(t)\,dt$

Using substitution method we obtain the following standard integrals.

- $\int \tan x dx = \log|\sec x| + C$
- $\int \cot x dx = \log|\sin x| + C$
- $\int \sec x dx = \log|\sec x + \tan x| + C$
- $\int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + C$

- **Integrals of some particular functions are as follows:**

- $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$
- $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$
- $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
- $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$
- $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$
- $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$

- **Integration by Partial Fractions:**
A rational function is defined as the ratio of two polynomials in the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x and $Q(x) \neq 0$. If $\frac{P(x)}{Q(x)}$ is improper, then $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$ where T(x) is a polynomial in x and $\frac{P_1(x)}{Q(x)}$ is a proper rational function. Assume we want to evaluate $\int \frac{P(x)}{Q(x)}dx$, where $\frac{P(x)}{Q(x)}$ is a proper rational function. It is possible to write the integrand as a sum of simpler rational functions by partial fraction decomposition as follows:

- $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$, $a \neq b$
- $\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$
- $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
- $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
- $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$, where $x^2 + bx + c$ cannot be factorized further.

- **Integration by parts:**
If $f(x)$ and $g(x)$ are the two functions then,

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left[f'(x)\int g(x)dx\right]dx,$$

where $f(x)$ is the first function and $g(x)$ is the second function. It can be stated as follows:**"The integration of the product of two functions = (First function) x (integral of the second function) – Integral of [(differential coefficient of the first function) x (integral of the second function)]"**

- Integral of the type $\int e^x[f(x) + f'(x)]dx = e^x f(x) + C$
- Some special types of integrals are as follows:

- $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$
- $\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$

➢ $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

➢ Integrals of the types $\int \frac{dx}{ax^2 + bx + c}$ or $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$ can be transformed into standard form by expressing $ax^2 + bx + c$ $= a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$

➢ Integrals of the types $\int \frac{px + qdx}{ax^2 + bx + c}$ or $\int \frac{px + qdx}{\sqrt{ax^2 + bx + c}}$ can be transformed into standard form by expressing $px + q$ $= A\frac{d}{dx}\left(ax^2 + bx + c\right) + B$ $= A(2ax + b) + B$, where A and B are determined by comparing coefficients on both sides.

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

1 Mark Questions

1. Write the anti derivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)$.

[DELHI 2014]

2. Evaluate $\int (1 - x)\sqrt{x}dx.$

[ALL INDIA 2012]

3. Write the value of $\int \frac{dx}{x^2 + 16}$.

[DELHI 2011]

4. Write the value of $\int \sec x(\sec x + \tan x)$

[DELHI 2011]

5. Evaluate $\int \frac{(1 + \log x)^2}{x}dx$

[ALL INDIA 2011]

6. Find : $\int \frac{\sin^2 x - \cos^2 x}{sinxcosx}dx$

[ALL INDIA 2017]

2 Marks Questions

7. Find $\int \frac{dx}{x^2 + 4x + 8}$

[DELHI 2017]

8. Find : $\int \frac{dx}{5 - 8x - x^2}$

[ALL INDIA 2017]

9. $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx$

[DELHI 2018]

4 Marks Questions

10. Evaluate: $\int \frac{\sin(x - a)}{\sin(x + a)}dx$

[DELHI 2013]

11. Integrate the following w.r.t. x: $\frac{x^2 - 3x + 1}{\sqrt{1 - x^2}}$.

[DELHI 2015]

12. Evaluate: $\int \frac{x^2}{(x^2 + 4)(x^2 + 9)}dx$

[DELHI 2013]

13. Find $\int \frac{2x}{(x^2 + 1)(x^2 + 2)^2}dx$

[DELHI 2017]

[TOPIC 2] Properties of a Definite Integrals and Limit of a Sum

Summary

- **Definite Integrals:**

  A definite integral is denoted by $\int_a^b f(x)dx$, where a is called the lower limit of the integral and b is called the upper limit of the integral.

- **Definite integral as the limit of a sum:**

  The definite integral $\int_a^b f(x)dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis.This can be mathematically defined as following:

  $$\int_a^b f(x)\,dx = (b-a)\lim_{h\to 0}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right]$$

  Where, $h = \frac{b-a}{n} \to 0$ as $n \to \infty$ $\int_a^b f(x)\,dx$ is defined as the area function where the area of the region is bounded by the curve $y = f(x)$, $a \le x \le b$, the $x-axis$ and the ordinates $x = a$ and $x = b$. Let x be a given point in $[a, b]$. Then $\int_a^x f(x)\,dx$ represents the **Area function $A(x)$**.

- **First fundamental theorem of integral calculus:**

  Let f be a continuous function on the closed interval $[a, b]$ and let $A(x) = \int_a^b f(x)\,dx$ for all $x \ge a$ be the area function. Then $A'(x) = f(x)$, for all $x \in [a, b]$.

- **Second fundamental theorem of integral calculus:**

  Let f be continuous function defined on the closed interval $[a, b]$ and F be an anti derivative of f. Then

  $$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a).$$

- **Properties of Definite Integrals** are as follows:

  ➢ $P_0 : \int_a^b f(x)dx = \int_a^b f(t)\,dt$

  ➢ $P_1 = \int_a^b f(x)\,dx = -\int_b^a f(x)dx.$

  In particular $\int_a^a f(x)\,dx = 0$

  ➢ $P_2 : \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

  ➢ $P_3 : \int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

  ➢ $P_4 : \int_0^a f(x)dx = \int_0^a f(a-x)dx$

  ➢ $P_5 : \int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)\,dx$

  ➢ $P_6 : \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx,$ if $(2a - x) = f(x)$ and 0, if $(2a - x) = -f(x)$.

  ➢ $$P_7 : \int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, \text{ if f is an even function,} \\ \qquad \text{i.e., } f(-x) = f(x)(2a-x) = f(x) \\ 0, \text{ if f is an odd function,} \\ \qquad \text{i.e., if } f(-x) = -f(x) \end{cases}$$

Summary

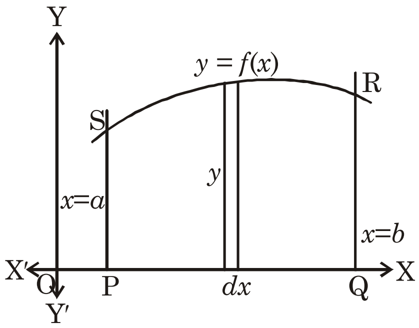
- **Area under simple curves**
  - ➢ Consider that a curve y = f(x), the line x = a, x = b and x-axis collectively acquires an area and the area under the curve is considered as composed of large number of vertical thin strips.

    Now assume that there is an arbitrary strip with height y and width dx.

    Then dA which represents area of elementary strip = ydx, where y = f(x).

    Total area A of the region between the curve y = f(x), x = a, x = b and x-axis is equal to the sum of areas of all elementary vertical thin strips across the region PQRS.

    which is given by, $A = \int_a^b y\,dx = \int_a^b f(x)dx$



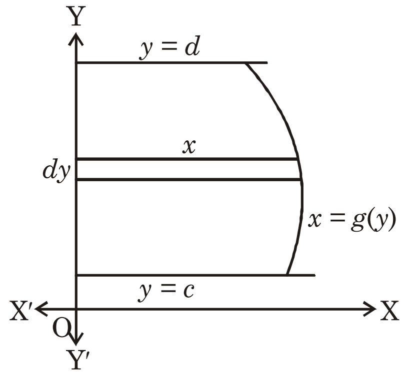
    Elementary Area: The area which is located at an arbitrary position within the region which is specified by some value of x between a and b

  - ➢ Now consider the area A of the region which is bounded by the curve x = g(y), the lines y = c, y = d and y-axis.

    Total area A of the region between the curve x = g(y), y = c, y = d and y-axis is equal to the sum of areas of all elementary horizontal thin strips.

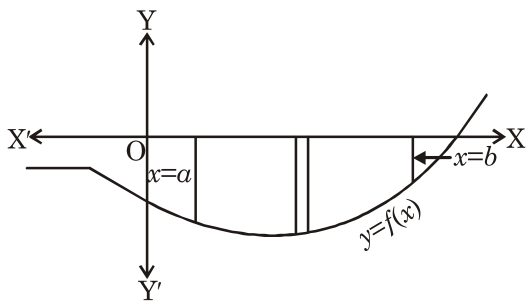
    In this case the area A is given by

$$A = \int_c^d x\,dy = \int_c^d g(y)dy$$



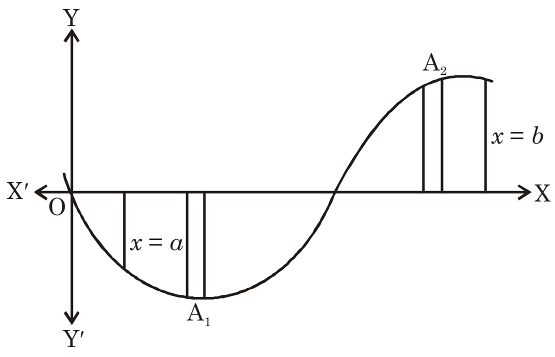
  - ➢ If the curve is positioned below x-axis, which is f(x) < 0 from x = a to x = b, then the numerical value of the area which is bounded by the curve y = f(x), x-axis and the ordinates x = a, x = b will come out to be negative. But, if the numerical value of the area is to be taken into consideration, then is given by:

$$A = \left|\int_a^b f(x)\,dx\right|$$



  - ➢ There is a possibilitythat some portion of the curve is located above x-axis and some portion of it is located below x-axis.

    Let suppose A_1 is the area below x-axis and $A_2$is the area above x-axis. Now, the area of the region which is bounded by the curve y = f(x), x = a, x = b and x-axis can be given by $A = |A_1| + |A_2|$.



- **Area of the region bounded by a curve and a line**
  - ➢ Area of the region bounded by a line and a curve is used to find the area bounded by a line and a parabola, a line and an ellipse, a line and a circle etc. The standard equation will be used for these mentioned curves.
  - ➢ Area of the region can be calculated by taking the sum of the area of either horizontal or vertical elementary strips but vertical strips are mostly preferred.
- **Area between two curves**
  - ➢ Assume that there are two curves, y = f(x) and y = g(x), where f(x) ≥ g(x) in [a, b]. The ordinates x = a and x = b give the point of intersection of these two curves. Suppose that these curves intersect at f(x) with width dx.

    Consider an elementary vertical strip of height y, where y = f(x).

    $\therefore \qquad dA = y\,dx$

    Now the area is given by,

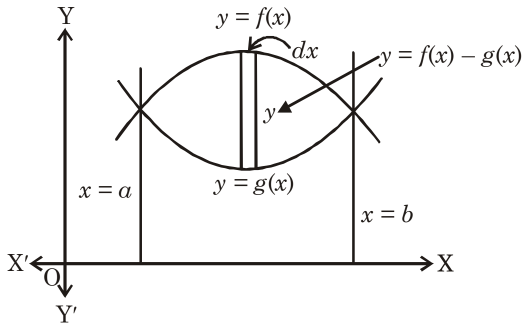
$$A = \int_a^b [f(x) - g(x)]dx = \int_a^b f(x)\,dx - \int_a^b g(x)dx$$

    which can be stated as,

    A = Area bounded by the curve {y = f(x)}

    – Area bounded by the curve {y = g(x)}

    where f(x) > g(x).



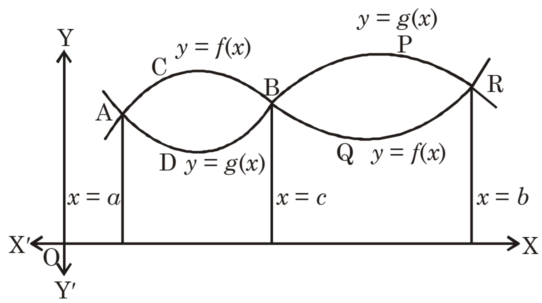
  - ➢ In other case if the two curves y = f(x) and y = g(x) where f(x) ≤ g(x) in [a, c] and f(x) ≤ g(x) in [c, b] with a condition that a < c < b, intersect at x = a, x = c and x = b, then the area bounded by the curves is given by:

$$A = \int_a^c |f(x) - g(x)|dx + \int_c^b |g(x) - f(x)|dx$$

    which is stated as:

    Total area = Area of the region ACBDA

    + Area of the region BPRQB



[TOPIC 1] Formation of Differential Equations

Summary

- **Differential Equation:** Differential equation is an equation which involves derivative of the dependent variable with respect to independent variable. Here $x\frac{dy}{dx}+y=0$ is the example of differential equation.
- General notations for derivatives are:

$$\frac{dy}{dx}=y', \frac{d^2y}{dx^2}=y'', \frac{d^3y}{dx^3}=y'''$$

or $\frac{d^ny}{dx^n}=y_n$

- **Order of a differential equation:** The order of the highest order derivative in any given differential equation is called the order of the differential equation.

Example: $\frac{dy}{dx}-\sin x=0$ has the order 2.

- **Degree of a differential equation:** If a differential equation involves a polynomial equation in its derivative, then its degree can be defined as the highest power of the highest order derivative in it.

Example: $\frac{dy}{dx}-\sin x=0$ has the degree 1.

Degree (if defined) and order are always positive.

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

1 Mark Questions

1. Find the differential equation representing the curve $y = cx + c^2$.

[ALL INDIA 2015]

2. Write the differential equations representing the family of curves $y = mx$, where m is an arbitrary constant.

[ALL INDIA 2013]

3. Write the degree of the differential equation

$$x^3\left(\frac{d^2y}{dx^2}\right)^2+x\left(\frac{dy}{dx}\right)^4=0.$$

[DELHI 2013]

4. Find the differential equation representing the family of curves $\upsilon=\frac{A}{r}+B,$ where A and B are arbitrary constants

[DELHI 2015]

2 Marks Question

5. Find the differential equation representing the family of curves $y=ae^{bx+5}$, where a and b are arbitrary constants.

[DELHI 2018]

4 Marks Questions

6. Find the differential equation for all the straight lines, which are at a unit distance from the origin.

[ALL INDIA 2015]

[TOPIC 2] Solution of Different Types of Differential Equations

- **Solution of a differential equation:** Solution of a differential equation is a function satisfying that differential equation.
  - **General Solution:** The solution having the arbitrary constants (equal to the order of the differential equation) is called a general or primitive solution of the differential equation.
  - **Particular Solution:** The solution which does not have the arbitrary constants is called a particular solution. It is acquired by substituting the particular values in the arbitrary constants.
- If the general solution of any differential equation is given, then the function is to be differentiated successively (as many times as the total number of arbitrary constants) in order to form the differential equation and then the arbitrary constants are eliminated.
- A differential equation which can be separated completely such that the terms which contains x can be written with dx and that of containing y with dy, can be solved with the help of variable separable method. The solution of such equations are of the form $\int f(x)dx = \int g(x)dx + C$, where C is an arbitrary constant.
- **Homogeneous Differential Equation:** A differential equation of the form $\frac{dy}{dx} = f(x, y)$ is called as homogeneous differential equation if f (x, y) is a homogeneous function of degree 0.
- To solve a homogeneous equation of the form $\frac{dy}{dx} = f(x, y) = g\left(\frac{y}{x}\right)$, substitutions are made as $\frac{y}{x} = v$ or y = vx and then general solution is found out by solving $\frac{dy}{dx} = v + x\frac{dv}{dx}$.
- To solve a homogeneous equation of the form $\frac{dx}{dy} = f(x, y)$, substitutions are made as $\frac{x}{y} = v$ or x = vy and then general solution is found out by writing $\frac{dx}{dy} = v + y\frac{dv}{dy}$.
- **Linear Differential Equation:** A differential equation which can be expressed as $\frac{dy}{dx} + Py = Q$, where P and Q are the constants or functions of x only, is called a linear differential equations of first order.
- To solve a linear differential equation, it is first written as $\frac{dy}{dx} + Py = Q$, then the integrating factor is found as $\text{I.F.} = e^{\int P dx}$. After that, the solution is given by $y(\text{I.F.}) = \int (Q \times \text{I.F.})dx + C$.
- If the linear differential equation is of the form $\frac{dx}{dy} + Px = Q$ (P and Q are constants or functions of y only), then $\text{I.F.} = e^{\int P dy}$ and the solution is given by $x(\text{I.F.}) = \int (Q \times \text{I.F.})dy + C$.

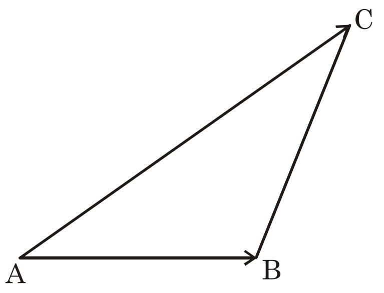
[TOPIC 1] Algebra of Vectors

Summary

- A vector quantity has both magnitude and direction where the magnitude is a distance between the initial and terminal point of the vector. Let's assume a vector starts at a point A and ends at a point B. Therefore the magnitude of the vector is denoted by $|\overrightarrow{AB}|$.
- $\overrightarrow{OA} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector of any point $A(x, y, z)$ having a magnitude equal to $\sqrt{x^2 + y^2 + z^2}$. Where O is the origin (0, 0, 0) and P is any point in the space.
- The angles α, β, γ are known as the direction angles which are made by the position vector and the positive x, y, z – axes respectively and their cosine values ($\cos\alpha$, $\cos\beta$, $\cos\gamma$) are known as direction cosines, denoted by l, m, n respectively.
- The projections of a vector along the respective axes are represented by the direction ratios which are the scalar components of the vector. They are denoted by a, b, c respectively.
- The direction cosines, direction ratios, and magnitude of a vector are related as:

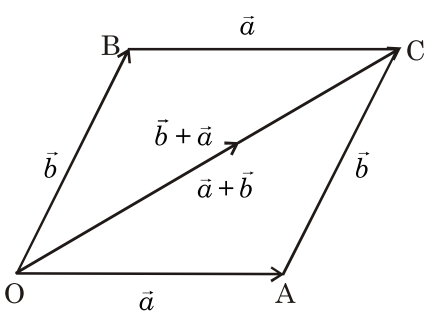
$$l = \frac{a}{r}, m = \frac{b}{r}, n = \frac{c}{r}$$

- In general, $l^2 + m^2 + n^2 = 1$ but $a^2 + b^2 + c^2 \neq 1$.
- Zero vector (also known as a null vector) is symbolized by $\vec{0}$. Its initial and terminal points coincide.
- A unit vector has a magnitude equal to 1 and is denoted by $\hat{a}$.
- If two or more than two vectors have the same initial points, they are called as co-initial vectors.
- The vectors which are parallel to the same line are known as collinear vectors.
- The vectors having equal magnitude and same direction are called equal vectors.
- A vector having the same magnitude as the given vector but opposite direction is known as the negative of the given vector.
- **Triangle law of vector addition:** Let's say that A, B, and C are the vertices of a triangle then



$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

- **Parallelogram law of vector addition:** If two vectors are represented by the two adjacent sides of a parallelogram, then their sum is represented by the diagonal of that parallelogram through their common point. For example, the



$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$

$\Rightarrow \overrightarrow{OC} = \vec{a} + \vec{b}$

- Vector addition is commutative as well as associative in nature and also has zero vector as an additive identity.
- The multiplication of any vector $\vec{a}$ by a scalar λ is denoted by $\lambda\vec{a}$ and has the same direction as the original vector if λ is positive and opposite direction if λ is negative. Its magnitude is $|\lambda\vec{a}| = |\lambda||\vec{a}|$.

- The unit vector of any vector $\vec{a}$ in its direction is written as $\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$
- The unit vectors along the positive x, y, z axes are denoted by $\hat{i}, \hat{j}, \hat{k}$ respectively.
- **Component form of a vector:** The component form of any vector is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ where x, y, z are called as the scalar components and $x\hat{i}, y\hat{j}, z\hat{k}$ as the vector components of $\vec{r}$. x, y, z is also called the rectangular components.
- If two vectors are in their component form as $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then
  - The sum of the vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$.
  - The difference of the vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$
  - The vectors $\vec{a}$ and $\vec{b}$ are equal if $a_1 = b_1, a_2 = b_2$ and $a_3 = b_3$.
  - The multiplication of a vector $\vec{a}$ by scalar λ is given by $\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$.
- **Vector joining two points:** The magnitude of a vector $\overrightarrow{A_1A_2}$ joining two points $\overrightarrow{A_1}(x_1, y_1, z_1)$ and $\overrightarrow{A_2}(x_2, y_2, z_2)$ is

  $\overrightarrow{A_1A_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- **Section Formula:** The position vector of a point C dividing the line segment joining two points A and B (having position vectors $\vec{a}, \vec{b}$ respectively) in the ratio of $m : n$
  - Internally: $\vec{r} = \frac{m\vec{b} + n\vec{a}}{m + n}$
  - Externally: $\vec{r} = \frac{m\vec{b} - n\vec{a}}{m - n}$
  - If C is the midpoint of A and B, then $\vec{r} = \frac{\vec{a} + \vec{b}}{2}$

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

1 Mark Questions

1. If $\vec{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ and $\vec{b} = 3\hat{i} - y\hat{j} + \hat{k}$ are two equal vectors, then write the value of x + y + z

   [DELHI 2013]

2. If a unit vector $\vec{a}$ makes angles $\frac{\pi}{3}$ with $i, \frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ *with* $\hat{k}, [0,1]$ then find the value of θ.

   [DELHI 2013]

3. Find the Cartesian equation of the line which passes through the point (–2, 4, –5)and is the parallel to the line $\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}$.

   [DELHI 2013]

4. If a line makes angles 90°.60°and θ with x, y and z-axis repectively, where θ is acute, then find θ.

   [DELHI 2015]

5. If a line makes angle 90° and 60° respectively with the positive direction of x and y axes, find the angle which it makes with the positive direction of z – axis.

   [DELHI 2017]

6. Find the vector equation of the line passing through the point $A(1,2,-1)$ and parallel to the line $5x - 25 = 14 - 7y = 35z$.

   [DELHI 2017]

[TOPIC 2] Product of Two Vectors, Scalar Triple Product

Summary

- **Scalar (or dot) product of two vectors $\vec{a}$ and $\vec{b}$:** The scalar product of two vectors having an angle θ between them is denoted by $\vec{a}\cdot\vec{b}$ and is defined by

  $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$

  $\Rightarrow \cos\theta=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

  or $\theta=\cos^{-1}\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$

- If $\vec{a}\cdot\vec{b}=0 \Leftrightarrow \vec{a}\perp\vec{b}$
- Properties of scalar product:
  - Let $\vec{a},\vec{b}$ and $\vec{c}$ be any three vectors then $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$
  - Let $\vec{a}$ and $\vec{b}$ be any two vectors, and λ be any scalar. Then $(\lambda\vec{a})\cdot\vec{b}=\lambda(\vec{a}\cdot\vec{b})=\vec{a}\cdot(\lambda\vec{b})$
- Projection of a vector $\vec{a}$ on another vector $\vec{b}$ is given as $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)$.
- Projection of a vector $\vec{b}$ on another vector $\vec{a}$ is given as $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)$.
- **Vector (or cross) product of two vectors $\vec{a}$ and $\vec{b}$:** The vector product of two vectors having an angle θ between them is denoted by $\vec{a}\times\vec{b}$ and is defined by:

  $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\hat{n}$

  $\sin\theta=\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

- If $\vec{a}\times\vec{b}=\vec{0}\Leftrightarrow\vec{a}\parallel\vec{b}$
- If the vectors are in their component form as $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then their cross product is given by

  $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

  The dot product is given by

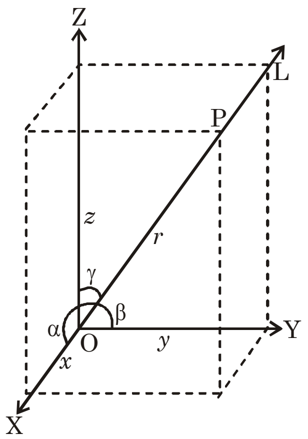
  $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3$
- Properties of vector product:
  - Let $\vec{a},\vec{b}$ and $\vec{c}$ be any three vectors then $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$.
  - Let $\vec{a}$ and $\vec{b}$ be any two vectors, and λ be any scalar. Then $(\lambda\vec{a})\times\vec{b}=\lambda(\vec{a}\times\vec{b})=\vec{a}\times(\lambda\vec{b})$
- Scalar triple product of three vectors $[A, B, C]$ is given by $\vec{A}.(\vec{B}\times\vec{C})$ which is the volume of a parallelepiped whose sides are given by vectors $\vec{A},\vec{B}$ and $\vec{C}$.

[TOPIC 1] Direction Cosines and Lines

Summary

- **Direction Cosines:**

  These are the cosines of the angles made by the line with the positive directions of the coordinate axes.



The direction cosines of line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are

$$\pm\frac{x_2-x_1}{AB}, \pm\frac{y_2-y_1}{AB}, \pm\frac{z_2-z_1}{AB} \text{ where}$$

$$AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Assume the direction cosines of the line are l, m and n and that the line is passing through the point $A(x_1, y_1, z_1)$, then the equation of the line is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

Also, $l^2 + m^2 + n^2 = 1$

- **Direction Cosines:**

  Direction ratios are any three numbers which are proportional to the direction cosines.

  Let the direction ratios be a, b and c, then

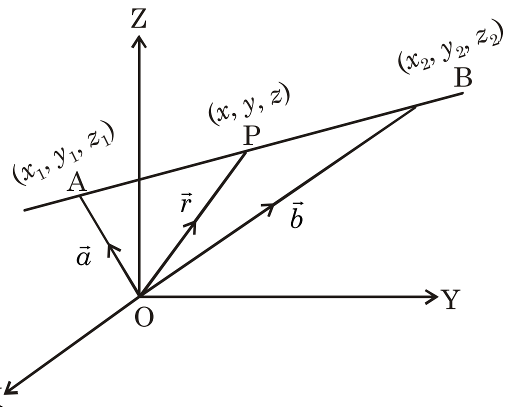
$$l=\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m=\pm\frac{b}{\sqrt{a^2+b^2+c^2}} \text{ and}$$

$$n=\pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

- **Lines:**

  ➤ Equation of the line which passes through two given points:

  Assume that the position vectors of $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are $\vec{a}$ and $\vec{b}$ respectively.



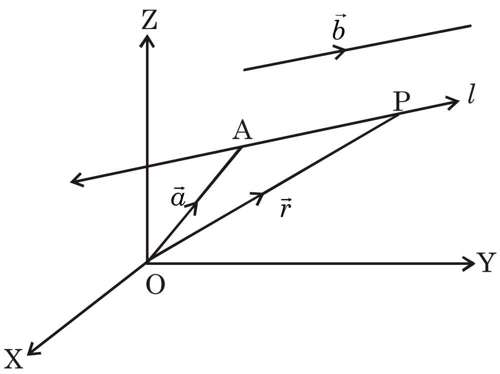
Then, the vector equation of the line is

$$\vec{r}=\vec{a}+k\left(\vec{b}-\vec{a}\right), k\in\mathbb{R}$$

The equation of the line in Cartesian form is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

➤ Equation of the line which passes through a given point having a given direction



The vector equation of the line is $\vec{r}=\vec{a}+\lambda\vec{b}$

The equation of the line in Cartesian form when it passes through $A(x_1, y_1, z_1)$ is

$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ where the direction ratios are a, b and c.

The angle between the lines $\vec{r} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{r} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ is given by

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

The shortest distance between the lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ is given by

$$\left|\frac{\left(\vec{b_1}\times\vec{b_2}\right)\cdot\left(\vec{a_2}-\vec{a_1}\right)}{\left|\vec{b_1}\times\vec{b_2}\right|}\right|$$

The shortest distance between the lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is given by

$$\left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2 + (c_1a_2-c_2a_1)^2 + (a_1b_2-a_2b_1)^2}}\right|$$

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

1 Mark Questions

1. If the Cartesian equations of a line are $\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4}$, Write the vector equation for the line.

[ALL INDIA 2014]

2. If a line has direction ratios 2, -1, -2, then what are its direction cosines?

[ALL INDIA 2018]

3. Write the direction cosines of the vector $-2\hat{i} + \hat{j} - 5\hat{k}$.

[DELHI 2011]

4. What are the direction cosines of a line, which makes equal angles with the coordinate axes?

[ALL INDIA 2014]

2 Marks Question

5. The x-coordinate of a point lies on the line joining the points $P(2, 2, 1)$ and $Q(5, 1, -2)$ is 4. Find its z-coordinate.

[ALL INDIA 2017]

4 Marks Questions

6. Show that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ intersect. Also find their point of intersection.

[DELHI 2014]

7. Find the shortest distance between the lines:

$$\vec{r} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(\hat{i} - 3\hat{j} + 2\hat{k}\right)$$

$$\vec{r} = \left(4\hat{i} + 5\hat{j} + 6\hat{k}\right) + \mu\left(2\hat{i} + 3\hat{j} + \hat{k}\right)$$

[ALL INDIA 2011]

8. The scalar product of the vector $\vec{a} = \left(\hat{i} + \hat{j} + \hat{k}\right)$ with a unit vector along the sum of vector $\vec{b} = \left(2\hat{i} + 4\hat{j} - 5\hat{k}\right)$ and $\vec{c} = \left(\lambda\hat{i} + 2\hat{j} + 3\hat{k}\right)$ is equal to one. Find the value of λ and hence find the unit vector along $\vec{b} + \vec{c}$.

[ALL INDIA 2014]

9. Find the coordinates of the foot of perpendicular drawn from the point $A(-1, 8, 4)$ to the line joining the points $B(0, -1, 3)$ and $C(2, -3, -1)$. Hence find the image of the point A in the line BC.

[ALL INDIA 2016]

[TOPIC 2] Plane

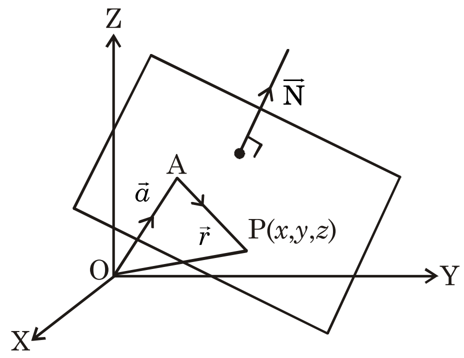
Summary

- **Plane**

  A surface so that when the two points are taken on it, the line segment lies joining the two points lies on the surface is called a plane.

  The equation of the plane is $\vec{r}\cdot\hat{n} = d$ where $\hat{n}$ is the unit vector normal to plane of origin. The equation of the plane in normal form is $lx + my + nz = d$ where l, m, n are direction cosines.

- **Equation of a plane perpendicular to a given vector**

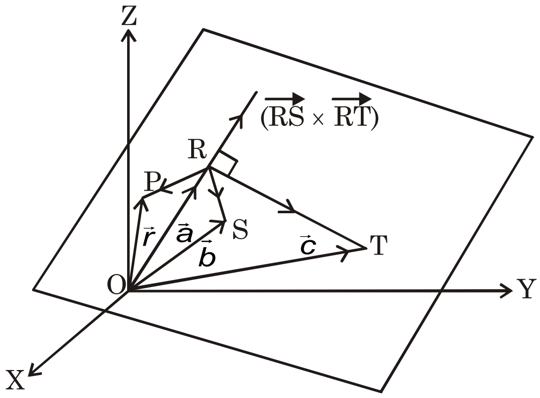


  The equation of a plane through a point whose position vector is $\vec{a}$ and perpendicular to the vector $\vec{N}$ is $\left(\vec{r}-\vec{a}\right)\cdot\vec{N} = 0$

  Equation of a plane perpendicular to a given vector and passing through a given point is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

- **Equation of a plane passing through three non collinear points**

  Let the non collinear points be $R(x_1, y_1, z_1)$, $S(x_1, y_2, z_2)$, $T(x_3, y_3, z_3)$ and $\vec{r}$ be the position vector.



  Equation of a plane passing through three non collinear points is

$$\left(\vec{r}-\vec{a}\right)\cdot\left[\left(\vec{b}-\vec{a}\right)\times\left(\vec{c}-\vec{a}\right)\right] = 0$$

  In Cartesian plane,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

  The equation of the plane in the intercept form is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ where a, b, c are x, y, z intercepts respectively.

- **The plane passing through intersection of two given lines**

  It has the equation

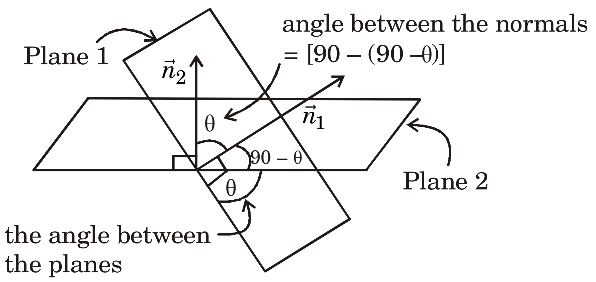
$$\vec{r}\cdot\left(\vec{n_1}+\lambda\vec{n_2}\right) = d_1 + \lambda d_2$$

  In Cartesian system,

$$(A_1x + B_1y + C_1z - d_1) + l(A_2x + B_2y + C_2z - d_2) = 0$$

- **Angle between two planes**

  The angle between the planes is given by the angle between their normals.



Let the angle between the planes be θ.

$$\cos\theta = \left|\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right|$$ where $\vec{n}_1, \vec{n}_2$ are normal to the planes.

In Cartesian form, $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$

$$\cos\theta = \left|\frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}\right|$$

The distance between a plane $Ax + By + Cz + D$ and the point (x_1, y_1, z_1) is given by

$$\left|\frac{A_1x + B_1y + C_1z + D_1}{\sqrt{A^2 + B^2 + C^2}}\right|$$

The angle between the line $\vec{r} = \vec{a} + \lambda\vec{b}$ and the plane $\vec{r}\cdot\hat{n} = d$ is $\sin\varphi = \left|\frac{\vec{b}\cdot\hat{n}}{|\vec{b}||\hat{n}|}\right|$

PREVIOUS YEARS' EXAMINATION QUESTIONS

TOPIC 2

1 Mark Questions

1. Find the distance between the planes $2x - y + 2z = 5$ and $5x - 2.5y + 5z = 20$.

   [ALL INDIA 2017]

2. Write the sum of intercepts cut off by the plane $\vec{r}.\left(2\hat{i} + \hat{j} - \hat{k}\right) - 5 = 0$ on the three axes.

   [ALL INDIA 2016]

3. Find the acute angle between the plane $5x - 4y + 7z - 13 = 0$ and the y-axis

   [ALL INDIA 2015]

4. Find the length of the perpendicular drawn from origin to the $2x - 3y + 6z + 21 = 0$ plane .

   [ALL INDIA 2013]

5. Find the vector equation of a plane which is at a distance of 5 units from the origin and its normal vector $2\hat{i} - 3\hat{j} + 6\hat{k}$

   [DELHI 2016]

6. Find λ, if the vectors $\vec{a} = \hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} - \hat{k}$ and $\vec{c} = \lambda\hat{j} + 3\hat{k}$ are coplanar.

   [DELHI 2015]

4 Marks Questions

7. Find the distance between the point $(-1, -5, -10)$ and the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$.

   [DELHI 2015]

8. Show that the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if $\vec{a} + \vec{b}, \vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar.

   [DELHI 2016]

9. Find the equation of the plane determined by the points $A(3, -1, 2)$, $B(5, 2, 4)$ and $C(-1, -1, 6)$ and hence find the distance between the plane and the point $P(6, 5, 9)$.

   [ALL INDIA 2012]

10. Show that the four points A, B, C and D with position vectors $4\hat{i} + 5\hat{j} + \hat{k}, -\hat{j} - \hat{k}$, $3\hat{i} + 9\hat{j} + 4\hat{k}$ and $4(-\hat{i} + \hat{j} + \hat{k})$ are respectively coplanar.

    [ALL INDIA 2014]

Summary

- **Linear Programming**

  Linear programming is a method which provides the optimization (maximization or minimization) of a linear function composed of certain variables subject to the number of constraints.

- **Applications of Linear Programming**
  - Used in finding highest margin, maximum profit, minimum cost etc.
  - Used in industry, commerce, management science etc.

- **Linear Programming Problem (LPP)**

  Linear Programming problem is a type of problem in which a linear function z is maximized or minimized on certain conditions that are determined by a set of linear inequalities with non-negative variables.

- **Mathematical Formulation of LPP**
  - **Optimal value:** Maximum or Minimum value of a linear function
  - **Objective Function:** The function which is to be optimized (maximized/minimized).
  - **Linear objective function:** $Z = ax + by$ is a linear function form, where a, b are constants, which has to be maximized or minimized is called a linear objective function.

    For example- $Z = 340x + 60y$, where variables x and y are called **decision variables**.

- **Constraints:** The limitations as disparities on the factors of a LPP are called constraints. The conditions $x \geq 0$, $y \geq 0$ are called **non-negative restrictions**.

- **'Linear'** states that all mathematical relations used in the problem are linear relations. **Programming** refers to the method of determining a particular program or plan of action.

- **Mathematical Formulation of the Problem**

  A general LPP can be stated as (Max/Min) $Z = c_1x_1 + c_2x_2 + \dots\dots + c_nx_n$ subject to given constraints and the non-negative restrictions.
  - $x_1, x_2, \dots\dots, x_n \geq 0$ and all are variables.
  - $c_1, c_2, \dots\dots c_n$ are constants.

- **Graphical methods to solve a Linear Programming Problem**

- **Corner Point method:** This method is used to solve the LPP graphically by finding the corner points.

  Procedure-

  (i) Replace the signs of inequality by the equality and consider each constraint as an equation.

  (ii) Plotting each equation on the graph that will represent a straight line.

  (iii) The common region that satisfies all the constraints and the non-negative restrictions is known as the **feasible region**. It is a convex polygon.

  (iv) Determining the vertices of the convex polygon. These vertices of the polygon are also known as the extreme points or corners of the feasible region.

  (v) Finding the values of Objective function at each of the extreme points. Now, finding the point at which the value of the objective function is optimum as that is the optimal solution of the given LPP.

- **General features of a LPP**
  - The feasible region is always a convex region.
  - The maximum (or minimum) solution of the objective function occurs at the vertex (corner) of the feasible region.
  - If two corner points produce the same optimum (maximum or minimum) value of the objective function, then every point on the line segment joining these points will also give the same optimum (maximum or minimum) value.

- **Different Types of Linear Programming Problems**
  - **Manufacturing problems**

    In order to make maximum profit, determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed manpower, machine hours, warehouse space per unit of the output etc., in order to make maximum profit.
  - **Diet problems**

    Determining the minimum amount of different nutrients which should be included in a diet so as to minimize the cost of the diet.
  - **Transportation problems**

    To find the cheapest way of transporting a product from factories situated at different locations to different markets.
  - **Allocation problems**

    These problems are concerned with the allocation of a particular land/area of a company or any organization by choosing a certain number of employees and a certain amount of area to complete the assignment within the required deadline, given that a single person works on only one job within the assignment.
- Integration is the inverse of differentiation. Instead of diff

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

4 Marks Questions

1. A small firm manufactures necklace and bracelets. The total number of necklaces and bracelets that it can handle per day is at most 24. It takes 1 hour to make a bracelet and half an hour make a necklace. The maximum number of hours available per day is 16. If the profit on a necklace is Rs100 and that on a bracelet is Rs300, formulate an L.P.P for finding how many of each should be produced daily to maximize the profit? It is being given that at least one of each must be produced.

[DELHI 2017]

2. Solve the following L.P.P. graphically:

   Minimise $Z = 5x + 10y$

   Subject to $x + 2y \leq 120$

   Constraints: $x + y \geq 60$, $x - 2y \geq 0$ and x,y ≥ 0

[DELHI 2017]

3. A cooperative society of farmers has 50 hectares of land to grow two crops A and B The profit from crops A and B per hectare are estimated as Rs. 10,500 B and Rs. 9,000 respectively. To control weeds a liquid herbicide has to be used for crops A and B at the rate of 20 litres and 10 litres per hectare, respectively. Further not more that 800 litres of herbicide should be used in order to protect fish and wildlife is more important drainage from this land. Keeping in mind that the protection of fish and other wildlife is more important than earing profit ,how much land should be allocated to each crop so as to maximize the total profit ? From an LPP from the above and solve it graphically.

[ALL INDIA 2013]

6 Marks Questions

4. A manufacturer produces two products A and B. Both the products are processed on two different machines. The available capacity of first machine is 12 hours and that of second machine is 9 hours per day. Each unit of product A requires 3 hours on both machines and each unit of product B requires 2 hours on first machine and 1 hour on second machine, Each unit of product A is sold at Rs 7 profit and that of B at a profit of Rs 4. Find the production level per day for maximum profit graphically.

[DELHI 2016]

[TOPIC 1] Conditional Probability and Independent Events

Summary

- **Probability:**

  Let S be the sample space and E be the event in an experiment.

  Then,

  $$\text{Probability} = P(E) = \frac{\text{Number of favourable event}}{\text{Total number of events}}$$

  $$= \frac{n(E)}{n(S)}$$

  Where, $0 \le n(E) \le n(s)$

  $\Rightarrow 0 \le P(E) \le 1$

  Hence, the probability of the occurrence of an event E is denoted by $P(E)$

  Now, $P(\overline{E}) = 1 - P(E)$ ($P(\overline{E})$ can also be written as $P(E')$)

  - If probability of any event is one, this does not depict the certainty of that event.
  - In similar way if the probability of any event is zero, this does not depict that the event will never occur.
  - **Mutually Exclusive Event:** The two events which cannot occur simultaneously are called mutually exclusive events.

    Let $B = \{1, 2, 3, 4, 5, 6\}$

    X = the event of occurrence of a number greater than 5 = {6}

    Y = the event of occurrence of an even number = {2, 4, 6}

    Here, events X and Y are not mutually exclusive because they can occur together when the number 6 comes up.
  - Independent Events: If the occurrence or non-occurrence of one event is unaffected by the occurrence or non-occurrence of other, these events are called independent events.

    Consider an example of drawing two marbles one by one with replacement from a jar containing 2 red marbles and 1 yellow marble

    Now assume, X = the event of occurrence of a red marble in first draw

    And Y = the event of occurrence of a yellow marble in second draw

    So, here the probability of occurrence Y is not affected by that of X.

    Hence, events X and Y are independent events.
  - **Exhaustive Events:** If the performance of random experiment always results in the occurrence of at least one of the given set of events, the set of those events will be known as exhaustive.
  - If their union is the total sample space
  - If event A, B and C are disjoint pairs i.e.,

    Consider an example of throwing a die, $A = \{1, 2, 3, 4, 5, 6\}$

    Now assume X = the event of occurrence of an multiple of 2 = {2, 4, 6}

    Y = the event of occurrence of the number not divisible by 2 = {1, 3, 5}

    Z = the event of occurrence of multiple of 3 = {3,6}

    Here X and Y are mutually exclusive but Y and Z are not.

- **Conditional Probability:**

  The probability of occurrence of event A when B has already been occurred is known as Conditional probability also called probability of occurrence of A w.r.t B.

  Some important formulae related to conditional probability

  - $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$, $B \neq \phi$ i.e., $P(B) \neq 0$

- $P(B|A)=\dfrac{P(A\cap B)}{P(A)}$, $A\neq\phi$ *i.e.,* $P(A)\neq 0$
- $P(\overline{A}|B)=\dfrac{P(\overline{A}\cap B)}{P(B)}$, $P(B)\neq 0$
- $P(A|\overline{B})=\dfrac{P(A\cap\overline{B})}{P(\overline{B})}$, $P(\overline{B})\neq 0$
- $P(\overline{A}|\overline{B})=\dfrac{P(\overline{A}\cap\overline{B})}{P(\overline{B})}$, $P(\overline{B})\neq 0$
- $P(A|B)+P(\overline{A}|B)=1$

Some formulae

- $P(A\cup B)=P(A)+P(B)-P(A\cap B)$ *i.e.,* $\mathrm{P}(A \text{ or } B)=P(A)+P(B)-P(A \text{ and } B)$
- $P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)-P(A\cap B\cap C)$
- $P(\overline{A}\cap B)=P(\text{only } B)=P(B-A)=P(B \text{ but not } A)=P(B)-P(A\cap B)$
- $P(A\cap\overline{B})=P(\text{only } A)=P(A-B)=P(\text{A but not } B)=P(A)-P(A\cap B)$
- $P(\overline{A}\cap\overline{B})=P(B-A)=P(\text{neither } A \text{ nor } B)=1-P(A\cup B)$

PREVIOUS YEARS'

EXAMINATION QUESTIONS

TOPIC 1

2 Marks Questions

1. A black and a red die are rolled together. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

[DELHI 2018]

2. Prove that if E and F are independent events, then the events E and F ' are also independent.

[DELHI 2017]

3. A die, whose faces are marked 1, 2, 3 in red and 4, 5, 6 in green, is tossed. Let A be the event "number obtained is even" and B be the event "number obtained is red". Find if A and B are independent events.

[ALL INDIA 2017]

4 Marks Questions

4. Find the mean number of heads in three tosses of a fair coin

[ALL INDIA 2011]

5. A and B throw a pair of dice alternately, till one of them gets a total of 10 and wins the game. Find their respective probabilities of winning, if A starts first.

[ALL INDIA 2016]

6. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls? Given that

(i) the youngest is a girl

(ii) at least one is a girl.

[DELHI 2014]

7. Probabilities of solving a specific problem independently by A and B are $\frac{1}{2}$ *and* $\frac{1}{3}$ respectively if both try to solve the problem independently, find the probability that (i) the problem is solved (2) exactly one of them solves the problem.

[DELHI 2011]

[TOPIC 2] Baye's Theorem and Probability Distribution

Summary

- **BAYES' theorem:**
  - If E_1, E_2, E_3 ………. E_n are n non-empty constituting a partition of sample space S i.e., S_1, S_2, S_3 ……. S_n are pair wise disjoint and $E_1 \cup E_2 \cup E_3 \cup \ldots\ldots \cup E_n = S$ and A is any event of non- zero probability, then

$$P(E \mid A) = \frac{P(\mathrm{E}_i).P(A \mid \mathrm{E}_i)}{\sum_{j=1}^{n} P(\mathrm{E}_j)P(A \mid \mathrm{E}_j)}, i = 1,2,3,\ldots\ldots.n$$

  - For example,

$$P(E_1 \mid A) = \frac{P(\mathrm{E}_1).P(A \mid \mathrm{E}_1)}{P(\mathrm{E}_1).P(A \mid \mathrm{E}_1) + P(\mathrm{E}_2).P(A \mid \mathrm{E}_2) + P(\mathrm{E}_3).P(A \mid \mathrm{E}_3)},$$

$i = 1,2,3,\ldots\ldots.n$

  - It is also known as the formula for the probability of cause.
  - Prior probabilities are the probabilities which are known before the experiment takes place.
  - $P(A \mid E_n)$ are called posterior probabilities.
- **Random Variable:**

  A real valued function defined over the sample space of an experiment is known as random variable. It is denoted by uppercase letters X, Y, Z etc.

  - **Discrete random variable :** When only finite or countably infinite number of values can be taken by the random variable then it is called discrete random variable.
  - **Continuous random variable:** When any value between two given limits can be taken by the variable then it is called continuous random variable.

    If the values of a random variable together with the corresponding probability are known, then this is called the **probability distribution of the random variable**.
- **Formulae:**
  - Mean or Expectation of a random variable

    $X = X = \mu = \sum_{i=1}^{n} x_i P_i$

  - Variance $= \left(\sigma^2\right) = \sum_{i=1}^{n} P_i {x^2}_i - \mu^2$
  - Standard deviation $= \sigma = \sqrt{\text{Variance}}$
- **Bernoulli Trials:**

  They are basically known as trials of a random experiment.

  If they satisfy the following conditions:
  - There should be a finite number of trials.
  - The trials should be independent.
  - Each trial has exactly two outcomes: success or failure.
  - The probability of success remains the same in each trial
- **Binomial distribution:**

  A Binomial distribution with probability of success in each trial as p and with n Bernoulli trials is denoted by $B(n, p)$

  n and p are the parameters of Binomial Distribution

  Therefore the expression $P(x = r)$ or $P(r)$ is called the probability function of Binomial Distribution.