Solution

SECTION A

1. Given differential equation.

$$x^2 \frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^4$$

∴ Order of differential equation = Order of the highest derivative

$= 2$ (½)

and degree of differential equation = Power of $\frac{d^2y}{dx^2}$

$= 1$ (½)

2. Given f(x) = x + 7 and g(x) = x – 7

We know that

fog(x) = f(g(x))

= f(x – 7)

= x – 7 + 7

= x (½)

Therefore, $\frac{d}{dx}(fog)(x) = \frac{d}{dx}(x) = 1$ (½)

Given : $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

3. $\because 2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2\times 1 & 2\times 3 \\ 2\times 0 & 2\times x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 6 \\ 0 & 2x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2+y & 6+0 \\ 0+1 & 2x+2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (2+y) & 6 \\ 1 & (2x+2) \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

(½)

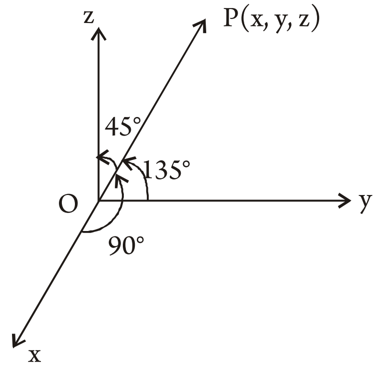
On comparing both the matrices,

$\Rightarrow 2 + y = 5 \therefore y = 5 - 2 = 3$

and $2x + 2 = 8 \Rightarrow 2x = 6 \therefore x = \frac{6}{2} = 3$

Hence, x – y = 3 – 3 = 0 (½)

4. Suppose $\alpha = 90°$, $\beta = 135°$ and $\gamma = 45°$



(½)

$\therefore\ l = \cos\alpha = \cos 90° = 0$

$m = \cos\beta = \cos 135° = -\frac{1}{\sqrt{2}}$ and $n = \cos\gamma = \cos 45° = \frac{1}{\sqrt{2}}$

Hence, required direction cosines of the line are $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$. (½)

OR

We know that the vector equation of line passing through A $(\vec{a})$ and parallel to the vector $(\vec{b})$ is given by

$\vec{r} = \vec{a} + \lambda\vec{b}$ where λ is a scalar (parameter) (½)

Given $\vec{a} = 3\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{b} = 2\hat{i} + 2\hat{j} - 3\hat{k}$

Hence, the equation of the line is

$\vec{r} = (3\hat{i} + 4\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 2\hat{j} - 3\hat{k})$ (½)

Section B

5. Here a, b $\in$ R

$\therefore$ (ab + 1) $\in$ R

So, * is a binary operation on R

[If for $\forall$ a, b $\in$ A and a * b $\in$ A, then * is a binary operation] (1)

For Associative Property :

Suppose a, b, c $\in$ R, then

we have to prove ,

a *(b * c) =(a * b)* c

L.H.S = a*(b * c) = a* (bc + 1)

= [a(bc + 1)] + 1

= abc + a + 1

Again, R.H.S. = (a* b)* c = (ab + 1)* c

= [(ab + 1)c] + 1

= abc + c + 1

$\therefore$ L.H.S $\neq$ R.H.S

Hence, * binary operation is not associative. (1)

6. Given, $A=\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}$

$\therefore\ A^2 = A.A$

$$=\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 8\\ 1 & -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}(2\times2+0\times2+1\times1) & (2\times0+0\times1-1\times1) & (2\times1+0\times3+1\times0)\\ (2\times2+1\times2+3\times1) & (2\times0+1\times1-1\times3) & (2\times1+1\times3+3\times0)\\ \left(1\times2+2\times(-1)+0\times1\right) & (1\times0-1\times1-0\times1) & (1\times1-1\times3+0\times0)\end{bmatrix}$$

$$=\begin{bmatrix}5 & -1 & 2\\ 9 & -2 & 5\\ 0 & -1 & -2\end{bmatrix}$$ (1)

and $5A=5\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}=\begin{bmatrix}10 & 0 & 5\\ 10 & 5 & 15\\ 5 & -5 & 0\end{bmatrix}$

Hence, $A^2-5A=\begin{bmatrix}5 & -1 & 2\\ 9 & -2 & 5\\ 0 & -1 & -2\end{bmatrix}-\begin{bmatrix}10 & 0 & 5\\ 10 & 5 & 15\\ 5 & -5 & 0\end{bmatrix}$ (½)

$$=\begin{bmatrix}(5-10) & (-1-0) & (2-5)\\ (9-10) & (-2-5) & (5-15)\\ (0-5) & (-1+5) & (-2-0)\end{bmatrix}=\begin{bmatrix}-5 & -1 & -3\\ -1 & -7 & -10\\ -5 & 4 & -2\end{bmatrix}$$ (½)

7. Suppose $I=\int\sqrt{1-\sin 2x}\,dx, \frac{\pi}{4}<x<\frac{\pi}{2}$

$=\int\sqrt{\cos^2 x+\sin^2 x-2\sin x\cos x}\,dx$ (½)

$[\because\ \sin 2x = 2\sin x\cos x]$

$=\int\sqrt{(\cos x-\sin x)^2}\,dx$

$=\int|\cos x-\sin x|\,dx$

$\therefore$ for $x\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ $\sin x > \cos x$

$\therefore\ |\cos x-\sin x| = \sin x-\cos x$ (½)

Hence, $I=\int(\sin x-\cos x)dx$

= – cos x – sin x + C (1)

where C is the constant of integration.

OR

Suppose $I=\int \sin^{-1}(2x)dx$

Let 2x = t

$\Rightarrow 2\,dx = dt \quad \therefore I=\frac{1}{2}\int 1.\sin^{-1}(t)dt$

On integrating by parts.

$$=\frac{1}{2}\left[(\sin^{-1}t)t-\int\frac{t\,dt}{\sqrt{1-t^2}}\right]$$ (1)

$$=\frac{1}{2}\left[t\sin^{-1}t-\frac{1}{2}\int\frac{2t\,dt}{\sqrt{1-t^2}}\right]$$

$$=\frac{1}{2}\left[t\sin^{-1}t+\frac{1}{2}\int\frac{-2t}{\sqrt{1-t^2}}dt\right]$$

$$=\frac{1}{2}\left[t\sin^{-1}t+\frac{1}{2}\times\frac{(1-t^2)^{\frac{-1}{2}+1}}{\frac{1}{2}}\right]+C$$

$$=\frac{1}{2}\left[t\sin^{-1}t+\sqrt{1-t^2}\right]+C$$

$$=\frac{1}{2}\times 2x\sin^{-1}(2x)+\frac{1}{2}\sqrt{1-4x^2}+C$$

$$=x\sin^{-1}(2x)+\frac{1}{2}\sqrt{1-4x^2}+C$$ (1)

where C is the constant of integration.

8. Given, $y = e^{2x}(a + bx)$

$\Rightarrow y=\frac{(a+bx)}{e^{-2x}}\Rightarrow e^{-2x}y=a+bx$ ...(i)

On differentiating equation (i) with respect to x,

$e^{-2x}\times\frac{dy}{dx}+y\times e^{-2x}\times(-2)=b$

$\Rightarrow e^{-2x}\left(\frac{dy}{dx}-2y\right)=b$ ...(ii) (1)

Again, on differentiating equation (ii) with respect to x,

$e^{-2x}\left(\frac{d^2y}{dx^2}-2\frac{dy}{dx}\right)+\left(\frac{dy}{dx}-2y\right)\times e^{-2x}\times(-2)=0$

$$\Rightarrow e^{-2x}\left(\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y\right)=0$$

$$\therefore \frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0$$ (1)

9. Suppose, the probability of success in single attempt is p.

$\therefore$ probability of getting an odd number = p

$$\Rightarrow p=\frac{3}{6}=\frac{1}{2}$$

and probability of unsuccessful attempt is

$$q = 1 - p = 1-\frac{1}{2}=\frac{1}{2}$$

(i) Probability of getting 5 success

$$= {}^6C_5 \times (p)^5 \times (q)^1$$

$$=6\times\left(\frac{1}{2}\right)^5\times\left(\frac{1}{2}\right)^1$$

$$=6\times\frac{1}{2^6}=\frac{6}{2^6}$$

$$=\frac{6}{64}=\frac{3}{32}$$ (1)

(ii) Probability of getting atmost 5 success

= 1 – Probability of getting 6 success

$$=1-{}^6c_6\times(P)^6$$

$$=1-1\times\left(\frac{1}{2}\right)^6$$

$$=1-\frac{1}{2^6}=1-\frac{1}{64}=\frac{63}{64}$$ (1)

OR

Given, $P(X=x)=\begin{cases} k, & \text{if } x=0 \\ 2k, & \text{if } x=1 \\ 3k, & \text{if } x=2 \\ 0, & \text{otherwise} \end{cases}$

we know that $\Sigma P(x_i) = 1$ (1)

$\therefore$ k + 2k + 3k = 1

$\Rightarrow$ 6k = 1

$\therefore k = \frac{1}{6}$ (1)

10. Given, A = "Number is ever"

$\Rightarrow$ A = {2, 4, 6} and B = "Number is marked red"

$\Rightarrow$ B = {1, 2, 3}

$\therefore$ A $\cap$ B = Numbers which is even as well as marked red

$= \{2, 4, 6\} \cap \{1, 2, 3\} = \{2\}$

So, Probability of occurrence of event A,

$$P(A) = \frac{3}{6} = \frac{1}{2}$$ (1)

and probability of occurrence of event B,

$$P(B) = \frac{3}{6} = \frac{1}{2} \text{ and } P(A \cap B) = \frac{1}{6}$$

$\therefore$ $P(A \cap B) \neq P(A).\ P(B)$

Hence, events A and B are not independent. (1)

11. Let two unit vectors are $\hat{a}$ and $\hat{b}$, then According to the equation

$\hat{a} + \hat{b} = \hat{r}$ $[\hat{r} \rightarrow \text{unit vector}]$

$$\Rightarrow |\hat{a} + \hat{b}| = |\hat{r}|$$ (½)

On squaring both sides,

$$\Rightarrow (\hat{a} + \hat{b}).(\hat{a} + \hat{b}) = \hat{r}.\hat{r}$$

$$\Rightarrow \hat{a}.\hat{a} + \hat{a}.\hat{b} + \hat{b}.\hat{a} + \hat{b}.\hat{b} = \hat{r}.\hat{r}$$

$$\Rightarrow 1 + \hat{a}.\hat{b} + \hat{a}.\hat{b} + 1 = 1 \quad [\because \hat{n}.\hat{n} = 1 \text{ and } \hat{a}.\hat{b} = \hat{b}.\hat{a}]$$

$$\Rightarrow 2\hat{a}.\hat{b} = 1 - 2 = -1$$

$$\therefore \hat{a}.\hat{b} = -\frac{1}{2} \quad \text{...(i)}$$ (½)

Magnitude of their difference $= |\hat{a} - \hat{b}|$

Let $|\hat{a} - \hat{b}| = t$

On squaring both sides,

$$t^2 = |\hat{a} - \hat{b}|^2 \Rightarrow t^2 = (\hat{a} - \hat{b}).(\hat{a} - \hat{b}) \quad [\because |\vec{a}|^2 = (\vec{a}).(\vec{a})]$$

$$= \hat{a}.\hat{a} - \hat{a}.\hat{b} - \hat{b}.\hat{a} + \hat{b}.\hat{b}$$

$$= 1 - 2\hat{a}.\hat{b} + 1 \quad [\because \hat{a}.\hat{a} = 1]$$ (½)

$$= 2 - 2\hat{a}.\hat{b}$$

$$= 2 - 2\left(-\frac{1}{2}\right) \quad [\text{using equation (i)}]$$

$$= 2 + 1 = 3$$

$$\therefore t = \sqrt{3}$$

$$\text{or } |\hat{a} - \hat{b}| = \sqrt{3}$$ (½)

Hence proved.

OR

Given, $\vec{a}=2\hat{i}+3\hat{j}+\hat{k}$

$\vec{b}=\hat{i}-2\hat{j}+\hat{k}$

and $\vec{c}=-3\hat{i}+\hat{j}+2\hat{k}$

$$\therefore \left[\vec{a}\,\vec{b}\,\vec{c}\right]=\vec{a}.(\vec{b}\times\vec{c})=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$=\begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \\ -3 & 1 & 2 \end{vmatrix}$$ (1)

$= 2(-4 - 1) -3(2 +3) + 1(1 - 6)$
$= 2 \times(-5)- 3 \times (5) + 1 \times (-5)$
$= -10 - 15 - 5$
$= - 30$ (1)

12. Let $I=\int \frac{\tan^2 x \, . \sec^2 x}{1-\tan^6 x} dx$

Let $\tan^3 x = t$
$\Rightarrow 3 \tan^2 x \sec^2 x \, dx = dt$ (½)

$$\therefore\ I=\int \frac{1}{3}\times\frac{dt}{1-t^2}$$

$$=\frac{1}{3}\int\frac{dt}{1-t^2}$$

$$=\frac{1}{3}\times\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|+C$$ (½)

Hence, $I=\frac{1}{6}\log\left|\frac{1+\tan^3 x}{1-\tan^3 x}\right|+C$ (1)

where C is constant of integration.

Section C

13. According to the question,

$\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}(2x) = \frac{\pi}{4} - \tan^{-1}(3x)$ (1)

$\Rightarrow \tan(\tan^{-1}(2x)) = \tan\left(\frac{\pi}{4}-\tan^{-1}(3x)\right)$

$$\Rightarrow 2x = \frac{\tan\frac{\pi}{4} - \tan\left(\tan^{-1}(3x)\right)}{1 + \tan\frac{\pi}{4} \times \tan\left(\tan^{-1}(3x)\right)}$$ (1)

$$\Rightarrow 2x = \frac{1-3x}{1+3x} \Rightarrow 2x + 6x^2 = 1 - 3x$$

$\Rightarrow 6x^2 + 5x - 1 = 0$

$\Rightarrow 6x^2 + 6x - x - 1 = 0$

$\Rightarrow 6x(x + 1) - 1(x + 1) = 0$

$\Rightarrow (x + 1)(6x - 1) = 0$

If $x + 1 = 0 \therefore x = -1$ [Invalid]

and if $6x - 1 = 0 \quad \therefore x = \frac{1}{6}$

Hence, $x = \frac{1}{6}$ (2)

14. Given,

$\log(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$

On differentiating with respect to x,

$$\frac{d}{dx}\left[\log\left(x^2 + y^2\right)\right] = \frac{d}{dx}\left[2\tan^{-1}\left(\frac{y}{x}\right)\right]$$ (1)

$$\Rightarrow \frac{1}{x^2+y^2} \times \frac{d}{dx}\left(x^2+y^2\right) = \frac{2}{1+\left(\frac{y}{x}\right)^2} \times \frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{1}{x^2+y^2}\left[2x + 2y\frac{dy}{dx}\right] = \frac{2x^2}{x^2+y^2}\left[\frac{x\frac{dy}{dx} - y}{x^2}\right]$$ (1)

$$\Rightarrow x + y\frac{dy}{dx} = x\frac{dy}{dx} - y$$

$$\Rightarrow (y - x)\frac{dy}{dx} = -y - x$$

$$\therefore \frac{dy}{dx} = \frac{-(x+y)}{-(x-y)} = \frac{x+y}{x-y}$$ (2)

Hence proved.

OR

$\because x^y - y^x = a^b$

On differentiating with respect to x,

$$\frac{d}{dx}\left(x^y\right) - \frac{d}{dx}\left(y^x\right) = 0$$ ...(i)

$x^y = v$ (let) $\quad \Rightarrow \quad \log v = y \log x$

On differentiating with respect to x,

$$\frac{1}{v}\frac{dv}{dx} = y\frac{d}{dx}(\log x) + \log x.\frac{dy}{dx}$$

$$\Rightarrow \frac{dv}{dx} = v\left[\frac{y}{x} + \log x.\frac{dy}{dx}\right]$$

$$\Rightarrow \frac{d}{dx}(x^y) = x^y\left[\frac{y}{x} + \log x.\frac{dy}{dx}\right] \quad ...(ii)$$ (1)

Similarly, let $y^x = u$

$$\Rightarrow \log u = x.\log y$$

On differentiating with respect to x.

$$\frac{1}{u}.\frac{du}{dx} = x\frac{d}{dx}(\log y) + \log y.1$$

$$\Rightarrow \frac{du}{dx} = u\left[\frac{x}{y}\frac{dy}{dx} + \log y\right]$$

$$\Rightarrow \frac{d}{dx}(y^x) = y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] \quad ...(iii)$$ (1)

Now, from equation (i).

$$x^y\left[\frac{y}{x} + \log x.\frac{dy}{dx}\right] - y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] = 0$$

$$\Rightarrow \frac{dy}{dx}\left[x^y.\log x - y^x\left(\frac{x}{y}\right)\right] = y^x.\log y - x^y\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^x.\log y - x^{y-1}.y}{x^y.\log x - y^{x-1}.x}$$ (2)

15. Let $I = \int \frac{3x+5}{x^2+3x-18}dx$

$$= \int \frac{\frac{3}{2}(2x+3) - \frac{9}{2} + 5}{x^2+3x-18}dx$$ (1)

$$= \frac{3}{2}\int \frac{2x+3}{x^2+3x-18}dx + \int \frac{5-\frac{9}{2}}{x^2+3x-18}dx$$

$$= \frac{3}{2}\int \frac{2x+3}{x^2+3x-18}dx + \frac{1}{2}\int \frac{1}{x^2+3x-18}dx$$

$$= \frac{3}{2}\int \frac{\frac{d}{dx}(x^2+3x-18)}{x^2+3x-18}dx + \frac{1}{2}\int \frac{dx}{\left(x+\frac{3}{2}\right)^2 - 18 - \frac{9}{4}}$$ (1)

$$=\frac{3}{2}\ln|x^2+3x-18|+\frac{1}{2}\int\frac{dx}{\left(x+\frac{3}{2}\right)^2-\left(\frac{9}{2}\right)^2}$$

$$=\frac{3}{2}\ln|x^2+3x-18|+\frac{1}{2}\times\frac{1}{2\times\frac{9}{2}}\log\left|\frac{x+\frac{3}{2}-\frac{9}{2}}{x+\frac{3}{2}+\frac{9}{2}}\right|+c$$

$$=\frac{3}{2}\ln|x^2+3x-18|+\frac{1}{18}\log\left|\frac{x-\frac{6}{2}}{x+\frac{12}{2}}\right|+c \quad (1)$$

$$=\frac{3}{2}\ln|x^2+3x-18|+\frac{1}{18}\log\left|\frac{2x-6}{2x+12}\right|+c$$

$$=\frac{3}{2}\ln|x^2+3x-18|+\frac{1}{18}\log\left|\frac{x-3}{x+6}\right|+c \quad (1)$$

16. To prove : $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$

$\therefore$ RHS $\int_0^a f(a-x)\,dx$

Suppose a – x = t $\Rightarrow$ – dx = dt
when x = 0, then t = a
and when x = a, then t = 0 (1)

$$=\int_0^a f(t)(-dt)$$

$$=\int_0^a f(t)\,dt \qquad \left[\because \int_a^b f(x)\,dx=-\int_b^a f(x)\,dx\right]$$

$$=\int_0^a f(x)\,dx \qquad \left[\because \int_a^b f(x)\,dx=-\int_b^a f(t)\,dt\right]$$

Hence, L.H.S. = R.H.S. (1)

Let $I=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$ ...(i)

$$\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$$

$$I=\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx \quad ...(ii)$$

On adding equation (i) and (ii).
we get,

$$2I=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx+\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx$$

$$2I=\int_0^{\pi}\frac{\pi \sin x}{1+\cos^2 x}dx \qquad (1)$$

$$\Rightarrow I=\frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$$

Let $\cos x = t \Rightarrow \sin x\, dx = -\, dt$

when $x = 0$, then $t = \cos 0 = 1$

and when $x = \pi$, then $t = \cos \pi = -1$

$$\therefore\ I=\frac{\pi}{2}\int_1^{-1}\frac{-dt}{1+t^2}$$

$$I=\frac{\pi}{2}\int_{-1}^{1}\frac{dt}{1+t^2}=\frac{\pi}{2}\left[\tan^{-1} t\right]_{-1}^{1}$$

$$=\frac{\pi}{2}\left[\tan^{-1}(1)-\tan^{-1}(-1)\right]$$

$$=\frac{\pi}{2}\left[\frac{\pi}{4}+\frac{\pi}{4}\right]$$

$$=\frac{\pi}{2}\times\frac{\pi}{2}=\frac{\pi^2}{4} \qquad (1)$$

17. Given, $\overrightarrow{OA}=\hat{i}+\hat{j}+\hat{k}$

$\overrightarrow{OB}=2\hat{i}+5\hat{j}$

$\overrightarrow{OC}=3\hat{i}+2\hat{j}-3\hat{k}$ and $\overrightarrow{OD}=\hat{i}-6\hat{j}-\hat{k}$

$\therefore\ \overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\left(2\hat{i}+5\hat{j}\right)-\left(\hat{i}+\hat{j}+k\right)$ (½)

$$=i+4\hat{j}-\hat{k}$$

and $\overrightarrow{CD}=\overrightarrow{OD}-\overrightarrow{OC}=\left(\hat{i}-6\hat{j}-\hat{k}\right)-\left(3\hat{i}-2\hat{j}-3\hat{k}\right)$

$$=-2\hat{i}-8\hat{j}+2\hat{k} \qquad (½)$$

We know that.

$$\cos\theta=\left|\frac{\overrightarrow{AB}.\overrightarrow{CD}}{|\overrightarrow{AB}|.|\overrightarrow{CD}|}\right| \quad \text{[where } \theta \text{ is the required angle]} \qquad (1)$$

$$=\left|\frac{\left(\hat{i}+4\hat{j}-\hat{k}\right).\left(-2\hat{i}-8\hat{j}+2\hat{k}\right)}{\sqrt{(1)^2+(4)^2+(1)^2}\times\sqrt{(-2)^2+(-8)^2+(2)^2}}\right|$$

$$=\left|\frac{-2-32-2}{\sqrt{18}\times\sqrt{72}}\right|$$

$$=\left|\frac{-36}{\sqrt{18}\times 2\sqrt{18}}\right|$$

$$=\frac{36}{2\times18}=\frac{36}{36}=1$$ (1)

$\Rightarrow \cos\theta = \cos 0$

$\therefore\ \theta = 0$

Therefore, lines $\overline{AB}$ and $\overline{CD}$ are collinear. (1)

18. Given $\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a)$

$$\therefore \text{ L.H.S} = \begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$$

On Applying $R_1 \to R_1 + R_2 + R_3$, (½)

$$\begin{vmatrix} (a+b+c-c-b) & (-c+a+b+c-a) & (-b-a+a+a+b+c) \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$$

On Applying $R_3 \to R_3 - R_1$ and $R_2 \to R_2 - R_1$.

$$= \begin{vmatrix} a & b & c \\ -(a+c) & (a+c) & -(a+c) \\ -(a+b) & -(a+b) & (a+b) \end{vmatrix}$$ (1)

$$=(a+b)(a+c)\begin{vmatrix} a & b & c \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$$ (½)

On Applying $R_3 \to R_3 + R_2$.

$$=(a+b)(a+c)\begin{vmatrix} a & b & c \\ -1 & 1 & -1 \\ -2 & 0 & 0 \end{vmatrix}$$ (1)

$= (a + b)\ (a + c)\ [-2(-b - c)]$

$= 2(a + b)\ (a + c)\ (b + c)$ (1)

Hence, L.H.S. = R.H.S.

19. Here, $y = \sin t$

On differentiating with respect to t,

$$\frac{dy}{dt} = \cos t \qquad ...(i)$$

Again, differentiating with respect to t,

$$\frac{d^2y}{dt^2} = -\sin t$$

Put $t = \frac{\pi}{4}$

$$\therefore \frac{d^2y}{dt^2} = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}} \qquad (1)$$

Given,

$$x = \cos t + \log\left[\tan\left(\frac{t}{2}\right)\right]$$

On differentiating with respect to t,

$$\Rightarrow \frac{dx}{dt} = -\sin t + \frac{\frac{1}{2}\sec^2\left(\frac{t}{2}\right)}{\tan\left(\frac{t}{2}\right)}$$

$$= -\sin t + \frac{1}{\sin t} \quad \left[\because 2\sin\frac{t}{2}\cos\frac{t}{2} = \sin t\right]$$

From equation (i) and (ii),

$$\frac{dy}{dt} \div \frac{dx}{dt} = \frac{\cos t}{-\sin t + \frac{1}{\sin t}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin t . \cos t}{1-\sin^2 t} = \frac{\sin t . \cos t}{\cos^2 t} = \frac{\sin t}{\cos t} = \tan t \qquad (1)$$

On differentiating with respect to x,

$$\frac{d^2y}{dx^2} = \sec^2 . \frac{dt}{dx}$$

$$= \frac{\sec^2 t}{\left(-\sin t + \frac{1}{\sin t}\right)} = \frac{\sec^2 t . \sin t}{1-\sin^2 t}$$

$$= \frac{\sin t}{\cos^4 t} \qquad (1)$$

Put $t = \frac{\pi}{4}$

$$\therefore \frac{d^2y}{dx^2} = \frac{\sin\frac{\pi}{4}}{\cos^4\left(\frac{\pi}{4}\right)} = \frac{\frac{1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^4} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^3}$$

$$= \left(\sqrt{2}\right)^3 = 2\sqrt{2} \qquad (1)$$

20. Here, $R = \{(a, b) : a \le b\}$

For reflexivity,

As $a \le a \forall a \in R$

$\because$ 'R' is reflexive. (1)

For symmetry,

suppose $(a, b) \in R$

$\therefore a \le b$

Here, it is not necessary that $b \le a$

$\therefore (b, a) \in R$ [False]

Therefore, 'R' is not symmetric. (2)

for transitivity,

Suppose aRb and bRc

$\therefore a \le b$ and $b \le c \Rightarrow a \le c$

$\therefore$ aRc

So, aRb, bRc $\Rightarrow$ aRc

$\therefore$ 'R' is transitive.

Hence, 'R' is reflexive as well as transitive but not symmetric (1)

OR

Given $f : N \to N$ defined as $f(x) = x^2 + x + 1$

Let $x_1, x_2 \in N$, then

$f(x_1) = f(x_2)$

$\Rightarrow {x_1}^2 + x_1 + 1 = {x_2}^2 + x_2 + 1 \Rightarrow (x_1^2 - x_2^2) + (x_1 - x_2) = 0 \Rightarrow (x_1 - x_2)[x_1 + x_2 + 1] = 0$

$\because x_1 + x_2 + 1 \ne 0$

$\therefore x_1 - x_2 = 0$

$x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$ (1½)

Therefore, 'f' is one-one and 'f' is not onto as $(x^2 + x + 1)$ does not attain all natural numbers for $x \in N$.

(½)

Hence, 'f' is one-one but not onto.

Now, $f(x) = x^2 + x + 1$

On putting $x = f^{-1}(x)$

$$\therefore x = (f^{-1}(x))^2 + f^{-1}(x) + 1$$

$$\Rightarrow (f^{-1}(x))^2 + f^{-1}(x) + (1 - x) = 0 \Rightarrow f^{-1}(x) = \frac{-1 \pm \sqrt{(1)^2 - 4(1-x)}}{2}$$

$$= \frac{-1 \pm \sqrt{1 - 4 + 4x}}{2} = \frac{-1 + \sqrt{4x - 3}}{2}$$

Hence, $f^{-1}(x) = \dfrac{-1 + \sqrt{4x-3}}{2}$ [$\because$ S contains natural numbers only.] (2)

21. According to the question,

equation of curve $y = \sqrt{3x - 2}$

On differentiating with respect to x,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{3x-2}} \times (3 - 0) = \frac{3}{2\sqrt{3x-2}}$$ (½)

Now, slope of tangent at point (x_1, y_1) is

$\frac{dy}{dx(x_1,y_1)}=\frac{3}{2\sqrt{3x_1-2}}$ and slope of line 4x – 2y + 50 is 2.

$$\therefore \frac{3}{2\sqrt{3x_1-2}}=2$$ (½)

$$\Rightarrow 3=4\sqrt{3x_1-2}$$

On squaring both sides,

$$\Rightarrow 9 = 16\,(3x_1 - 2)$$

$$\Rightarrow 3x_1-2=\frac{9}{16}\Rightarrow 3x_1=\frac{9}{16}+2$$

$$\Rightarrow 3x_1=\frac{41}{16}$$

$$\therefore x_1=\frac{41}{48}$$

$\because$ Point (x_1, y_1) lies on the curve

$$\therefore y_1=\sqrt{3\times\frac{41}{48}-2}=\sqrt{\frac{41-32}{16}}=\sqrt{\frac{9}{16}}=\frac{3}{4}$$ (1)

Therefore, equation of tangent,

$$\left(y-\frac{3}{4}\right)=2\left(x-\frac{41}{48}\right)$$

$$\Rightarrow y-\frac{3}{4}=2x-\frac{41}{24}\Rightarrow 2x-y=\frac{41}{24}-\frac{3}{4}$$

$$\Rightarrow 2x-y=\frac{41-18}{24}=\frac{23}{24}$$

$$\therefore 48x - 24y = 23$$

Now, equation of normal at the point of contact to the curve $\therefore\ y=\sqrt{3x-2}$

Suppose, slope of normal = m

$$\therefore m_1 m_2 = -1 \quad \Rightarrow m \times 2 = -1$$

$$\therefore m=-\frac{1}{2}$$ (1)

So, equation of normal at point $\left(\frac{41}{48},\frac{3}{4}\right)$

$$\Rightarrow\left(y-\frac{3}{4}\right)=-\frac{1}{2}\left(x-\frac{41}{48}\right)$$

$$\Rightarrow 2y-\frac{3}{2}=-x+\frac{41}{48}$$

$$\Rightarrow x+2y=\frac{41}{48}+\frac{3}{2}=\frac{41+72}{48}=\frac{113}{48}$$

$$\Rightarrow x+2y=\frac{113}{48}$$

$$\therefore\ 48x + 96y = 113$$ (1)

22. According to the given differential equation.

$$xdy - ydx = \sqrt{x^2 + y^2}\, dx$$

On dividing by dx,

$$\Rightarrow x\frac{dy}{dx} - y = \sqrt{x^2 + y^2}$$

$$\Rightarrow x\frac{dy}{dx} = \sqrt{x^2 + y^2} + y$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x^2 + y^2} + y}{x}$$

On putting y = vx and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, (1)

we get, $v + x\frac{dv}{dx} = \frac{\sqrt{x^2 + v^2 x^2} + vx}{x}$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{\sqrt{1 + v^2} + v}{1}$$

$$\Rightarrow x\frac{dv}{dx} = \sqrt{1 + v^2}$$

$$\Rightarrow \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}$$ (1)

On integrating both sides,
We get

$$\int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x}$$

$$\Rightarrow \log\left|v + \sqrt{1 + v^2}\right| = \log|x| + \log c$$

$$\Rightarrow \left|v + \sqrt{1 + v^2}\right| = |cx|$$

$$\Rightarrow \left|\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right| = |cx|$$

On putting x = 1 and y = 0

$|0 + \sqrt{1 + 0}| = |c|$

$\therefore$ c = 1

Hence, solution of the given differential equation is $y + \sqrt{x^2 + y^2} = x^2$. (2)

OR

According to the given differential equation,

$$(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$$

$$\Rightarrow (1+x^2)\frac{dy}{dx} = 4x^2 - 2xy$$

$$\Rightarrow \frac{dy}{dx} + \frac{2x}{(1+x^2)}y = \frac{4x^2}{(1+x^2)}$$ (1)

$\therefore$ This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where

$$P = \frac{2x}{1+x^2} \text{ and } Q = \frac{4x^2}{1+x^2}$$

$\therefore$ I.F. $= e^{\int P dx} = -e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} - 1 + x^2$ (½)

So, $y(1+x^2) = \int \frac{4x^2}{1+x^2}(1+x^2)\,dx + c$

$$\Rightarrow y(1+x^2) = \frac{4x^3}{3} + c$$ (1)

On putting x = 0 and y = 0

$\Rightarrow 0(0) = 0 + c$

$\therefore c = 0$ (½)

Hence, required solution of given differential equation is $y = \frac{4x^3}{3(1+x^2)}$ (1)

23. Given $L_1 : \frac{1-x}{3} = \frac{7y-14}{\lambda} = \frac{z-3}{2} \Rightarrow \frac{x-1}{-3} = \frac{y-2}{\frac{\lambda}{7}} = \frac{z-3}{2} = k_1 \text{(Let)}$

$\therefore$ Direction ratio's of $L_1 = \left\langle -3, \frac{\lambda}{7}, 2 \right\rangle$ (1)

Similarly,

$$L_2 \frac{7-7x}{3\lambda} = \frac{y-5}{1} = \frac{6-z}{5}$$

$$\Rightarrow \frac{x-1}{-\frac{3\lambda}{7}} = \frac{y-5}{1} = \frac{z-6}{-5} = k_2 \text{ (let)}$$

$\because$ Direction ratio's of $L_2 = \left\langle \frac{-3\lambda}{7}, 1, -5 \right\rangle$

$\because$ Lines L_1 and L_2 are perpendicular to each other

$$\therefore (-3)\left(-\frac{3\lambda}{7}\right) + \frac{\lambda}{7} \times 1 + 2 \times (-5) = 0$$

$$\Rightarrow \frac{9\lambda}{7} + \frac{\lambda}{7} - 10 = 0 \Rightarrow \frac{10\lambda}{7} = 10 \quad \therefore \lambda = 7$$ (1)

We know that lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ intersect each other if

$$\begin{vmatrix}(x_1-x_2) & (y_1-y_2) & (z_1-z_2)\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}=0$$

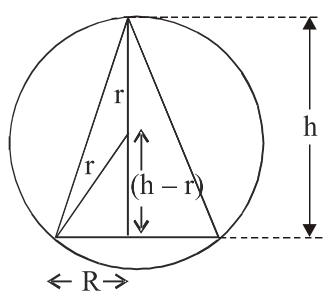
Now,

$$\Delta=\begin{vmatrix}(1-1) & (5-2) & (6-3)\\ -3 & 1 & 2\\ -3 & 1 & -5\end{vmatrix}=\begin{vmatrix}0 & 3 & 3\\ -3 & 1 & 2\\ -3 & 1 & -5\end{vmatrix}$$

$= -3\ (15 + 6) + 3\ (-3 + 3)$
$= -3 \times 21 + 0 = -63$ which is not equal to zero. Therefore, lines do not intersect. (2)

Section -D

24. Let the altitude of the cone is h unit, then, According to the question,
Here, Radius of base of the cone



$$R=\sqrt{r^2-(h-r)^2}$$
$$=\sqrt{r^2-h^2-r^2+2hr}$$
$$=\sqrt{2hr-h^2}$$

and volume of the cone $V=\frac{1}{3}\pi R^2h$

$$=\frac{\pi}{3}(2hr-h^2)h$$

$$=\frac{\pi}{3}(2h^2r-h^3)$$

$\because$ Volume of the cone is maximum,

$$\therefore \frac{dv}{dh}=0$$

$$\Rightarrow \frac{\pi}{3}[4hr-3h^2]=0$$

$\Rightarrow 4hr - 3h^2 = 0$

$\Rightarrow h(4r - 3h) = 0$

$\therefore\ h = 0 \text{ or } h = \frac{4R}{3}$ (2)

$\therefore$ For maximum volume, $\frac{d^2v}{dh^2}$ should be negative .

$$\frac{d^2v}{dh^2} = \frac{\pi}{3}[4r - 6h]$$

Put $h = \frac{4r}{3}$

$$\therefore \frac{d^2v}{dh^2} = \frac{\pi}{3}\left[4r - 6\left(\frac{4r}{3}\right)\right] = -\frac{4\pi r}{3} \text{ (negative)}$$ (2)

Therefore, for maximum volume, altitude of the cone should be $\frac{4r}{3}$.

$$\therefore v_{max} = \frac{\pi}{3}[2h^2r - h^3]$$

$$= \frac{\pi}{3}\left[2\left(\frac{4r}{3}\right)^2 r - \left(\frac{4r}{3}\right)^3\right]$$

$$= \frac{\pi}{3}\left[2 \times \frac{16}{9} \times r^3 - \frac{64}{27}r^3\right]$$

$$= \frac{\pi}{3}r^3\left[\frac{32}{9} - \frac{64}{27}\right]$$

$$= \frac{\pi}{3}r^3 \times \frac{(96-64)}{27}$$

$$= \frac{32}{81}\pi r^3$$ (2)

25. Given $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$\therefore |A| = 2(-4 + 4) + 3(-6 + 4) + 5(3 - 2)$

$= 2 \times 0 + 3 \times (-2) + 5 \times 1$

$= -6 + 5 = -1$ (1)

$$\text{Adj}(A) = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$ (1)

$$\therefore A^{-1} = \frac{1}{|A|}(\text{Adj}\,A) = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$ (1)

Again, given system of equations is

$2x - 3y + 5z = 11$

3x + 2y – 4z = –5

and x + y – 2z = –3

$$\therefore \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$ (1)

$$\Rightarrow A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$ (1)

$$= \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence, x = 1, y = 2 and z = 3 (1)

OR

We know that A = IA

$$\Rightarrow \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

From applying $R_1 \to R_1$.

$$\Rightarrow \begin{bmatrix} 1 & -1 & -2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$ (1)

From applying $R_2 \to R_2 - R_1$.

$R_3 \to R_3 - 3R_1$.

$$\Rightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & 3 & 5 \\ 0 & 4 & 7 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}A$$

From applying $R_2 \to -R_2 + R_3$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 4 & 7 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 1 \\ 3 & 0 & 1 \end{bmatrix}A$$ (1)

From applying $R_1 \to R_1 - R_2$.

$R_3 \to R_3 - 4R_2$.

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ -5 & 4 & -3 \end{bmatrix}A$$ (1)

From applying $R_3 \to -R_3$.

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 5 & -4 & 3 \end{bmatrix} A$$ (1)

From applying $R_2 \to R_2 - 2R_3$.

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix} A$$ (1)

$$\Rightarrow I = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix} A$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix}$$ (1)

26. Suppose,

Event E_1 = Operator A is chosen
Event E_2 = Operator B is chosen
Event E_3 = Operator C is chosen
Event E_3 = Operator C is chosen
and event K = Defective item is chosen

$$\therefore P(E_1) = \frac{50}{100}, P(E_2) = \frac{30}{100}, P(E_3) = \frac{20}{100}$$ (1)

Now $P\left(\frac{K}{E_1}\right) = \frac{1}{100}, P\left(\frac{K}{E_2}\right) = \frac{5}{100}, P\left(\frac{K}{E_3}\right) = \frac{7}{100}$ (2)

So, according to Baye's theorem,

$$P\left(\frac{E_1}{K}\right) = \frac{P(E_1).P\left(\frac{K}{E_1}\right)}{P(E_1).P\left(\frac{K}{E_1}\right) + P(E_2).P\left(\frac{K}{E_2}\right) + P(E_3).P\left(\frac{K}{E_3}\right)}$$ (1)

$$= \frac{\frac{50}{100} \times \frac{1}{100}}{\frac{50}{100} \times \frac{1}{100} + \frac{30}{100} \times \frac{5}{100} + \frac{20}{100} \times \frac{7}{100}}$$ (1)

$$= \frac{50}{50 + 150 + 140} = \frac{50}{340} = \frac{5}{34}$$

Hence, $P\left(\frac{E_1}{K}\right) = \frac{5}{34}$ (1)

27. Let A, B, C are the points with position vectors $\left(2\hat{i}+2\hat{j}-\hat{k}\right),\left(3\hat{i}+4\hat{j}+2\hat{k}\right)$ and $\left(7\hat{i}+6\hat{k}\right)$ respectively, then the required plane passes through the point A(2, 2, –1) and is normal to vector $\vec{n}$,

$\therefore\ \vec{n}=\overline{AB}\times\overline{AC}$

Now, $\overline{AB}=\left(3\hat{i}+4\hat{j}+2\hat{k}\right)-\left(2\hat{i}+2\hat{j}-\hat{k}\right)$

$=\hat{i}+2\hat{j}+3\hat{k}$

and $\overline{AC}=\left(7\hat{i}+6\hat{k}\right)-\left(2\hat{i}+2\hat{j}-\hat{k}\right)$

$=5i-2\hat{j}+7\hat{k}$ (2)

$$\therefore\ \vec{n}=\overline{AB}\times\overline{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ 5 & -2 & 7\end{vmatrix}$$

$=\hat{i}(14+6)+j(15-7)+\hat{k}(-2-10)=20i+8\hat{j}-12\hat{k}$ (1)

Therefore, required plane is $\vec{r}.\vec{n}=\vec{a}.\vec{n}$

$\Rightarrow\vec{r}.\left(20\hat{i}+8\hat{j}-12\hat{k}\right)=\left(2\hat{i}+2\hat{j}-\hat{k}\right).\left(20\hat{i}+8\hat{j}-12\hat{k}\right)$

$= 40 + 16 + 12 = 68$

$\Rightarrow\vec{r}.\left(5\hat{i}+2\hat{j}-3\hat{k}\right)=17$ (1)

$\therefore$ Cartesian equation of the plane is

$\left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(5\hat{i}+2\hat{j}-3\hat{k}\right)=17$

$\Rightarrow 5x + 2y - 3z = 17$

Suppose, equation of plane passing through (4, 3, 1) and parallel to plane $5x + 2y - 3z = 17$ is given as $5x + 2y - 3z = d$

$\Rightarrow 5 \times 4 + 2 \times 3 - 3 \times 1 = d$

$\therefore\ d = 20 + 6 - 3 = 23$

Hence, vector equation is $\vec{r}.\left(5\hat{i}+2\hat{j}-3\hat{k}\right)=23$ (2)

OR

The plane passes through the point A(–1, 3, –4) and contains the line $\frac{x-1}{1}=\frac{y-1}{2}=\frac{z}{-1}$ which passes through the point B(1, 10) and is parallel to the vector $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$.

$\therefore$ Required plane passes through point A(–1, 3, –4) and B(1, 1, 0) and is parallel to vector $\vec{b}$.

(1)

Now, $\vec{n}=\overline{AB}\times\vec{b}$ where $\vec{n}\rightarrow$ normal vector

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -2 & 4\\ 1 & 2 & -1\end{vmatrix}$$ (1)

$= \hat{i}(2-8)+\hat{j}(4+2)+\hat{k}(4+2)$

$= -6\hat{i}+6\hat{j}+6\hat{k}$ (1)

$\because$ Required plane passes through $\vec{\alpha} = -\hat{i}+3\hat{j}-4\hat{k}$ and is perpendicular to $\vec{x} = -6\hat{i}+6\hat{j}+6\hat{k}$.

$\therefore$ Its vector equation is

$(\vec{r}-\vec{\alpha}).\vec{x} = 0$

$\Rightarrow \vec{r}\,\vec{x} = \vec{\alpha}.\vec{x}$ (1)

$\Rightarrow \vec{r}.\left(-6\hat{i}+6\hat{j}+6\hat{k}\right) = \left(-\hat{i}+3\hat{j}-4\hat{k}\right).\left(-6\hat{i}+6\hat{j}+6\hat{k}\right)$

$\Rightarrow \vec{r}.\left(-\hat{i}+\hat{j}+\hat{k}\right) = \left(-\hat{i}+3\hat{j}-4\hat{k}\right).\left(-\hat{i}+\hat{j}+\hat{k}\right)$

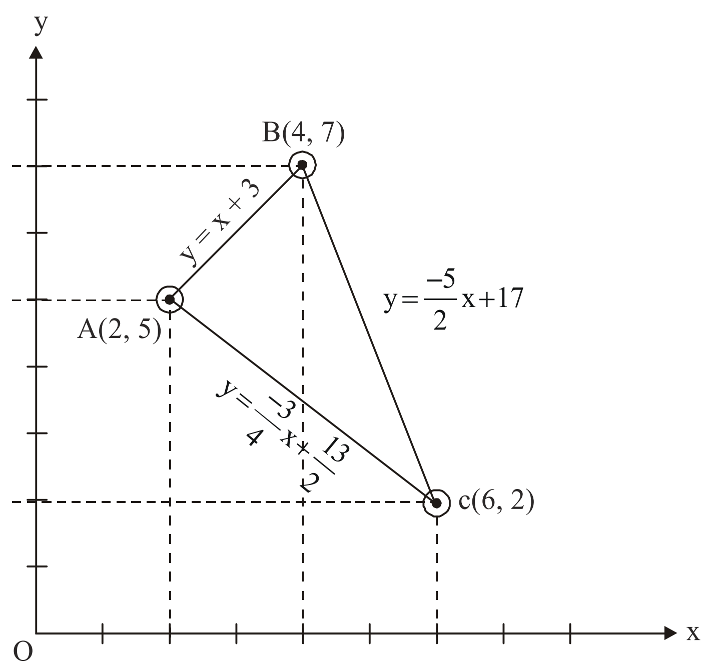
$\therefore \vec{r}.\left(-\hat{i}+\hat{j}+\hat{k}\right) = 0$

Hence, required length of perpendicular drawn from point (2, 1, 4) is

$$d = \left|\frac{\left(2\hat{i}+\hat{j}+4\hat{k}\right).\left(-\hat{i}+\hat{j}+\hat{k}\right)}{\sqrt{(-1)^2+(1)^2+(1)^2}}\right| = \left|\frac{-2+0+4}{\sqrt{3}}\right| = \sqrt{3} \text{ unit}$$ (2)

28. According to the question,

Equation of line AB,



(1)

$$(y-5) = \frac{7-5}{4-2}(x-2) \Rightarrow (y-5) = \frac{2}{2}(x-2)$$

$$\Rightarrow y - 5 = x - 2$$

$$\Rightarrow y = x + 3$$

Equation of line BC,

$$(y-7)=\frac{2-7}{6-4}(x-4) \Rightarrow y-7=\frac{-5}{2}(x-4) \Rightarrow y-7=\frac{-5x}{2}+10 \Rightarrow y=\frac{-5x}{2}+17$$

and equation of line CA,

$$(y-5)=\frac{2-5}{6-2}(x-2)$$

$$\Rightarrow y-5=\frac{-3}{4}(x-2) \Rightarrow y-5=\frac{-3x}{4}+\frac{3}{2}$$

$$\Rightarrow y=\frac{-3x}{4}+\frac{13}{2}$$ (2)

$$\therefore \text{ Area of } \Delta ABC=\int_2^4 (x+3)\,dx+\int_4^6\left(\frac{-5x}{2}+17\right)dx-\int_2^6\left(\frac{-3x}{4}+\frac{13}{2}\right)dx$$

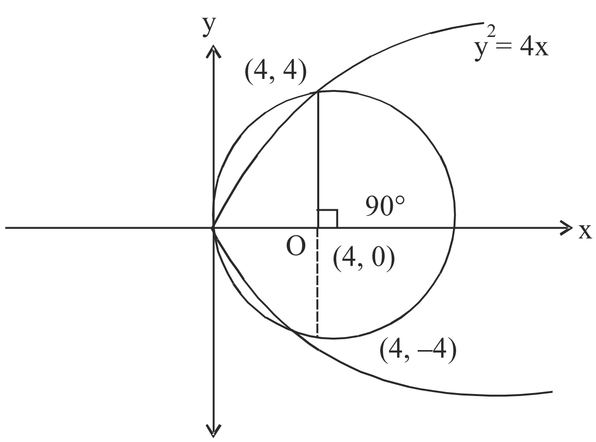
$$=\left[\frac{x^2}{2}+3x\right]_2^4+\left[\frac{-5x^2}{4}+17x\right]_4^6+\left[\frac{3x^2}{8}-\frac{13}{2}x\right]_2^6$$ (2)

$$=(8+12-2-6)+(-45+102+20-68)+\left(\frac{27}{2}-39-\frac{3}{2}+13\right)$$ (1)

= 12 + 9 – 14 = 7 unit²

OR

According to the question,



(2)

$\because x^2 + y^2 - 8x = 0$ ...(i)

$\therefore$ center of circle = (4, 0) and radius of circle = 4

And $y^2 = 4x$ ...(ii)

$\therefore$ Intersection points of circle $x^2 + y^2 - 8x = 0$ and parabola $y^2 = 4x$

$\Rightarrow x^2 + 4x - 8x = 0 \Rightarrow x^2 - 4x = 0$

$\Rightarrow x^2 = 4x$ (1)

$\therefore$ x = 0 and x = 4

Therefore, intersection points are (0, 0), (4, 4) and (4, –4)

$\therefore$ Required area $(A)=\int_0^4 2\sqrt{x}\,dx+\int_4^8\sqrt{8x-x^2}\,dx$ (1)

$$=\left[2\times\frac{2}{3}x^{\frac{3}{2}}\right]_0^4+\frac{1}{2}\left[(x-4)\sqrt{8x-x^2}+16\sin^{-1}\left(\frac{x-4}{4}\right)\right]_4^8$$

$$=\frac{4}{3}(8-0)+\frac{1}{2}\left[\frac{16\pi}{2}\right]$$

$$=\left(\frac{32}{3}+4\pi\right)\text{ unit}^2$$ (2)

29. Suppose, x items of model A and Y items of model B are made every day, then

Time constraints for skilled men,

$(2x+y)\le 5\times 8$

$\Rightarrow 2x+y\le 40$ ...(i) (1)

Time constraints for semi-skilled men,

$(2x+2y)\le 10\times 8$

$\Rightarrow 2x+3y\le 80$ ...(ii) (1)

and non-negative constraints,

$x\ge 0$ ...(iii)

and $y\ge 0$ ...(iv)

Objective function (profit) (1)

$z=15x+10y$

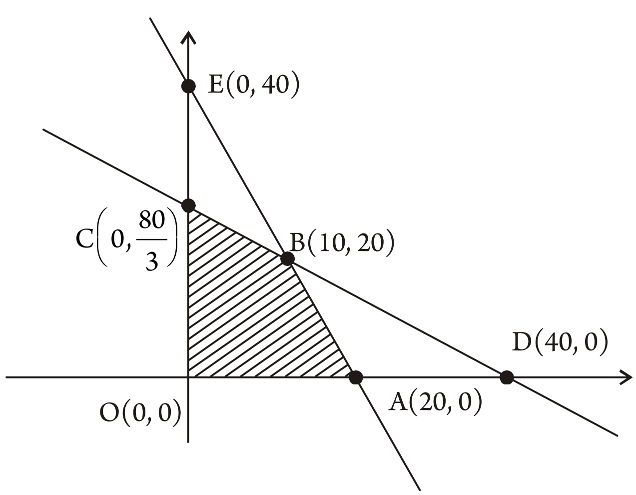
On solving the given problem graphically.

For, $2x+y=40$

x	0	20
y	40	0

and for, $2x+3y=80$

x	0	40
y	$\frac{80}{3}$	0



(2)

Shaded portion OABC is feasible region,

Points	O(0,0)	A(20,0)	B(10,20)	$C\left(0,\frac{80}{3}\right)$
Profit $z=15x+10y$	0	300	350	$\frac{800}{3}$

Therefore, at point B(10, 20), the profit z is maximum (350). Hence, for maximum profit, 10 items of model A and 20 items of model B should be made. (1)